

From  $V$  to  $F$  to  $KV$  following [AT].

$\log F = \Lambda[V][1] // \text{d}\sigma\{x, y\} \rightarrow \{y, x\};$   
 $\log F // \text{EulerE} // \text{adSeries}\left[\frac{\text{ad}_1}{\text{ad}}, \log F, \text{tb}\right]$

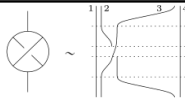
$$\begin{aligned} \overline{x} \rightarrow \text{LS} & \left[ \frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24} \overline{xyy}, -\frac{1}{180} \overline{xxxxy} + \frac{1}{80} \overline{xyxy} + \frac{1}{360} \overline{xyyy}, \right. \\ & -\frac{1}{720} \overline{xxxxxy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{720} \overline{xyxyxy} - \\ & \frac{\overline{xyyy}}{1440}, \frac{\overline{xxxxxy}}{5040} - \frac{\overline{xxxxy}}{1344} + \frac{13 \overline{xyxyy}}{15120} + \frac{1}{840} \overline{xyxyxy} + \\ & \left. \frac{\overline{xyxyxy}}{3360} + \frac{\overline{xyxyyy}}{6720} + \frac{\overline{xyxyyy}}{1260} + \frac{\overline{xyxyxy}}{1680} - \frac{\overline{xyyy}}{10080}, \dots \right], \\ \overline{y} \rightarrow \text{LS} & \left[ 0, \frac{\overline{xy}}{12}, \frac{1}{24} \overline{xyy}, -\frac{1}{360} \overline{xxxxy} + \frac{1}{120} \overline{xyxy} + \frac{1}{180} \overline{xyyy}, \right. \\ & -\frac{1}{720} \overline{xxxxxy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{720} \overline{xyxyxy} - \\ & \frac{\overline{xyyy}}{1440}, \frac{\overline{xxxxxy}}{10080} - \frac{\overline{xxxxy}}{2016} + \frac{\overline{xyxyy}}{1890} + \frac{\overline{xyxyxy}}{1120} + \frac{\overline{xyxyxy}}{5040} + \\ & \left. \frac{\overline{xyxyyy}}{2520} + \frac{1}{840} \overline{xyxyyy} + \frac{\overline{xyxyxy}}{1260} - \frac{\overline{xyyy}}{5040}, \dots \right] \end{aligned}$$

$\overline{\mathfrak{s}}[2, 1] = \overline{\mathfrak{s}}[3, 1] = \overline{\mathfrak{s}}[3, 2] = 0$ ; Solving for an associator  $\Phi$ .  
 $\overline{\mathfrak{s}}[3, 1, 2] = 1/24$ ;  $\overline{\mathfrak{s}} = \text{DKS}[3, \overline{\mathfrak{s}}]$ ;  
**SeriesSolve** $[\overline{\mathfrak{s}},$   
 $(\overline{\mathfrak{s}}^{\sigma[3,2,1]} \equiv -\overline{\mathfrak{s}}) \wedge$   
 $(\overline{\mathfrak{s}}^{**} \overline{\mathfrak{s}}^{\sigma[1,23,4]} ** \overline{\mathfrak{s}}^{\sigma[2,3,4]} \equiv \overline{\mathfrak{s}}^{\sigma[12,3,4]} ** \overline{\mathfrak{s}}^{\sigma[1,2,34]})];$   
 $\overline{\mathfrak{s}} (* \text{ Can raise degree to } 10 *)$

SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting  $\{\Phi[3, 1, 1, 2] \rightarrow 0\}$ .  
 SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting  $\{\Phi[3, 1, 1, 1, 1, 2] \rightarrow 0\}$ .

$$\begin{aligned} \text{DKS} & \left[ 0, \frac{1}{24} \overline{t_{13} t_{23}}, 0, -\frac{7 \overline{t_{13} t_{23} t_{23} t_{23}}}{5760} + \frac{7 \overline{t_{13} t_{13} t_{23} t_{23}}}{5760} - \frac{\overline{t_{13} t_{13} t_{13} t_{23}}}{1440}, \right. \\ & 0, \frac{31 \overline{t_{13} t_{23} t_{23} t_{23} t_{23}}}{967680} - \frac{157 \overline{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}}{1935360} - \\ & \frac{31 \overline{t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}}{387072} - \frac{31 \overline{t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}}{483840} + \\ & \frac{11 \overline{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}}{290304} + \frac{31 \overline{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}}{725760} + \frac{83 \overline{t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}}{967680} - \\ & \left. \frac{13 \overline{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}}{241920} + \frac{\overline{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}}{60480}, \dots \right] \end{aligned}$$

The "buckle"  $Z_B$ , from  $\Phi$ .



$R = \text{DKS}[t[1, 2] / 2];$   
 $Z_B = (-\overline{\mathfrak{s}})^{\sigma[13,2,4]} ** \overline{\mathfrak{s}}^{\sigma[1,3,2]} ** R^{\sigma[2,3]} ** (-\overline{\mathfrak{s}})^{\sigma[1,2,3]} **$   
 $\overline{\mathfrak{s}}^{\sigma[12,3,4]};$   
 $Z_B @ \{4\}$

$$\begin{aligned} \text{DKS} & \left[ \frac{\overline{t_{23}}}{2}, -\frac{1}{12} \overline{t_{13} t_{23}} - \frac{1}{24} \overline{t_{14} t_{24}} + \frac{1}{24} \overline{t_{14} t_{34}} + \frac{1}{12} \overline{t_{24} t_{34}}, \right. \\ & 0, \frac{\overline{t_{13} t_{23} t_{23} t_{23}}}{5760} + \frac{7 \overline{t_{14} t_{24} t_{24} t_{24}}}{5760} + \frac{\overline{t_{14} t_{34} t_{24} t_{24}}}{1920} - \\ & \frac{\overline{t_{14} t_{34} t_{34} t_{24}}}{1920} - \frac{7 \overline{t_{14} t_{34} t_{34} t_{34}}}{5760} - \frac{\overline{t_{24} t_{34} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{24} t_{34} t_{24}}}{1920} + \\ & \frac{\overline{t_{14} t_{24} t_{14} t_{34}}}{1920} - \frac{\overline{t_{14} t_{34} t_{24} t_{34}}}{1920} - \frac{1}{720} \overline{t_{13} t_{13} t_{23} t_{23}} + \\ & \frac{1}{720} \overline{t_{13} t_{13} t_{13} t_{23}} - \frac{7 \overline{t_{14} t_{14} t_{24} t_{24}}}{5760} + \frac{7 \overline{t_{14} t_{14} t_{34} t_{34}}}{5760} - \\ & \frac{\overline{t_{14} t_{24} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{14} t_{14} t_{24}}}{1440} - \frac{\overline{t_{14} t_{14} t_{14} t_{34}}}{1440} - \frac{1}{960} \overline{t_{14} t_{14} t_{24} t_{34}} + \\ & \left. \frac{\overline{t_{14} t_{24} t_{24} t_{34}}}{5760} - \frac{1}{960} \overline{t_{24} t_{24} t_{34} t_{34}} - \frac{\overline{t_{24} t_{24} t_{24} t_{34}}}{5760}, \dots \right] \end{aligned}$$

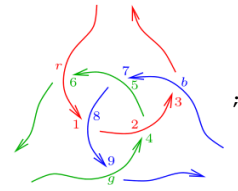
$V$  from  $Z_B$ , following [AET, BND].

$(\text{E1}[Z_B // \alpha\text{Map}[1, 2, 3, 4], \text{CWS}[0]] // \Gamma // \text{t}\eta^1 // \text{t}\eta^3 //$   
 $\text{h}\eta^2 // \text{h}\eta^4 // \text{h}\sigma\{3\} \rightarrow \{2\} // \text{t}\sigma\{2, 4\} \rightarrow \{1, 2\})[[$   
 $1]]$

$$\begin{aligned} 1 \rightarrow \text{LS} & \left[ 0, -\frac{\overline{12}}{24}, 0, \frac{\overline{71112}}{5760} - \frac{\overline{71122}}{5760} + \frac{\overline{1222}}{1440}, 0, \right. \\ & -\frac{31111112}{967680} + \frac{31111122}{483840} - \frac{83111222}{967680} - \frac{31112122}{725760} - \frac{31111212}{645120} + \\ & \frac{13112222}{241920} + \frac{10112122}{1451520} + \frac{527112212}{5806080} - \frac{\overline{122222}}{60480}, \dots \Big], \\ 2 \rightarrow \text{LS} & \left[ \frac{\overline{1}}{2}, -\frac{\overline{12}}{12}, 0, \frac{\overline{1112}}{5760} - \frac{1}{720} \overline{1122} + \frac{1}{720} \overline{1222}, \right. \\ & -\frac{\overline{11112}}{7680} + \frac{\overline{11122}}{3840} - \frac{\overline{11212}}{6912}, \\ & -\frac{\overline{111112}}{645120} + \frac{\overline{23111122}}{483840} - \frac{\overline{13111222}}{161280} - \frac{\overline{112122}}{22680} - \frac{\overline{41111212}}{580608} + \\ & \left. \frac{\overline{112222}}{15120} + \frac{\overline{121222}}{12096} + \frac{\overline{71112212}}{483840} - \frac{\overline{122222}}{30240}, \dots \right] \end{aligned}$$

The Borromean tangle.

$\text{Rs}[a_, b_] := \text{Es}[\langle a \rightarrow \text{LS}[0], b \rightarrow \text{LS}[\text{LW}@a] \rangle, \text{CWS}[0]];$   
 $\text{iRs}[a_, b_] := \text{Es}[\langle a \rightarrow \text{LS}[0], b \rightarrow -\text{LS}[\text{LW}@a] \rangle, \text{CWS}[0]];$   
 $\xi = \text{iRs}[r, 6] \text{Rs}[2, 4] \text{iRs}[g, 9] \text{Rs}[5, 7] \text{iRs}[b, 3] \text{Rs}[8, 1];$



$\text{Do}[\xi = \xi // \text{dm}[r, k, r], \{k, 1, 3\}];$   
 $\text{Do}[\xi = \xi // \text{dm}[g, k, g], \{k, 4, 6\}];$   
 $\text{Do}[\xi = \xi // \text{dm}[b, k, b], \{k, 7, 9\}];$   
 $\{\xi[[1]]_r @ \{5\}, \xi[[2]] @ \{5\}\} // \text{Print}$

$$\begin{aligned} \text{LS} & \left[ 0, \overline{bg}, \frac{1}{2} \overline{bbg} + \overline{bgr} + \frac{1}{2} \overline{bgg}, \right. \\ & \frac{1}{6} \overline{b b b g} + \frac{1}{2} \overline{b b g r} + \frac{1}{2} \overline{b g g r} + \frac{1}{4} \overline{b b g g} + \frac{1}{2} \overline{b g r r} + \frac{1}{6} \overline{b g g g}, \\ & \frac{1}{24} \overline{b b b b g} + \frac{1}{6} \overline{b b b g r} + \frac{1}{4} \overline{b b b g r} + \frac{1}{12} \overline{b b b g g} + \\ & \frac{1}{4} \overline{b b g r r} + \frac{1}{6} \overline{b g g g r} + \frac{1}{4} \overline{b g g r r} - \overline{b b g r g} + \\ & \frac{1}{12} \overline{b b g g g} - 2 \overline{b b r g g} + \frac{1}{6} \overline{b g r r r} + \frac{1}{2} \overline{b b g b g} - \\ & \overline{b g b r g} - \frac{1}{12} \overline{b b g b g} - \frac{1}{2} \overline{b g r g r} + \frac{1}{24} \overline{b g g g g}, \dots \Big], \\ \text{CWS} & \left[ 0, 0, 2 \overline{bgr}, \overline{bbgr} - \overline{bgbr} + \overline{bggr} - \overline{bgrg} + \overline{bgrr} - \overline{brgr}, \frac{\overline{bbgr}}{3} - \right. \\ & \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{3} - \\ & \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{3} + \frac{\overline{bbgr}}{2} - \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2}, \dots \Big] \end{aligned}$$

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.  
 [AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations*, Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067  
 [BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV*,  $\omega\epsilon\beta/\text{WKO1}$ ,  $\omega\epsilon\beta/\text{WKO2}$ ,  $\omega\epsilon\beta/\text{WKO3}$ ,  $\omega\epsilon\beta/\text{WKO4}$ , and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yetx2.

Warning. Fidgety!