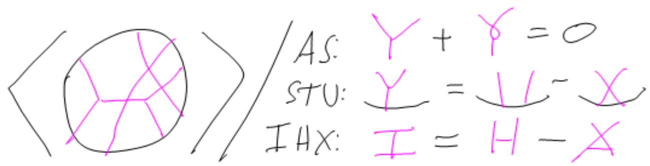
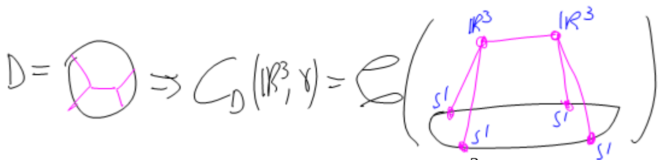
# Day 3: Chern-Simons, Gaussian Integration, Feynman Diagrams

**Recall.** $\mathcal{K}=\{\text{knots}\}$, $\mathcal{A}:=\text{gr}\mathcal{A}=\mathcal{D}/\text{rels}=$




Seek $Z\colon\mathcal{K}\to\hat{\mathcal{A}}$ such that if $K$ is $n$-singular, $Z(K)=D_k+\ldots$

$\mathcal{K}$ —(Z: high algebra; solving finitely many equations in finitely many unknowns)→ $\mathcal{A}:=\text{gr}\mathcal{K}$ —(given a "Lie" algebra $\mathfrak{g}$; low algebra: pictures represent formulas)→ "$\mathcal{U}(\mathfrak{g})$"



**Theorem.** Given a parametrized knot $\gamma$ in $\mathbb{R}^3$, up to renormalizing the "framing anomaly",

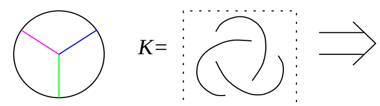
$$Z(\gamma)=\sum_{D\in\mathcal{D}}\frac{C(D)D}{|\text{Aut}(D)|}\int_{C_D(\mathbb{R}^3,\gamma)}\bigwedge_{e\in E(D)}\phi_e^*\omega\in\mathcal{A}$$

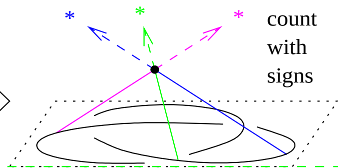
is an expansion. Here $\mathcal{D}$ is the set of all "Feynman diagrams", $E(D)$ is the set of internal edges (and chords) of $D$, $C_D(\mathbb{R}^3,\gamma)$ is the configuration space of placements of $D$ on/around $\gamma$, $\phi\colon C_D(\mathbb{R}^3,\gamma)\to(S^2)^{E(D)}$ is the "direction of the edges" map, and $\omega$ is a volume form on $S^2$.

## Cosmic Coincidences

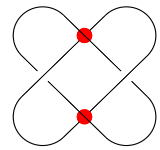
$\langle D,K\rangle_{\text{ш}}:=\begin{pmatrix}\text{The signed Stonehenge}\\\text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

$D=$ $K=$





The Gaussian linking number: $lk(\bigcirc\!\bigcirc)=\sum_{\text{vertical chopsticks}}(\text{signs})=\langle \bigcirc\!\!-\!\!\bigcirc,\ \bigcirc\!\bigcirc\rangle_{\text{ш}}$



C.F. Gauss

**The generating function of all cosmic coincidences:**

$$Z(K):=\lim_{N\to\infty}\sum_{\text{3-valent } D}\frac{\langle D,K\rangle_{\text{ш}}D}{2^c c!\binom{N}{e}}\in\mathcal{A}$$

D. Thurston

**Claim.** It all comes from the Chern-Simons-Witten theory,

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})}\mathcal{D}A\,\text{tr}_R hol_\gamma(A)\exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\text{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right],$$

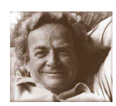
E. Witten

where $\Omega^1(\mathbb{R}^3,\mathfrak{g})$ is the space of all $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^3$ (really, connections), $k$ is some large constant, $R$ is some representation of $\mathfrak{g}$ and $\text{tr}_R$ is trace in $R$, and $hol_\gamma(A)$ is the holonomy of $A$ along $\gamma$.

**References.** Witten's *Quantum field theory and the Jones polynomial*, Axelrod-Singer's *Chern-Simons perturbation theory I-II*, D. Thurston's arXiv:math.QA/9901110, Polyak's arXiv:math.GT/0406251, and my videotaped 2014 class $\omega$/AKT.

**Gaussian Integration.** $(\lambda_{ij})$ is a symmetic positive definite matrix and $(\lambda^{ij})$ is its inverse, and $(\lambda_{ijk})$ are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let $\partial^i$ denote $\frac{\partial}{\partial t_i}$. Also let $C:=\frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

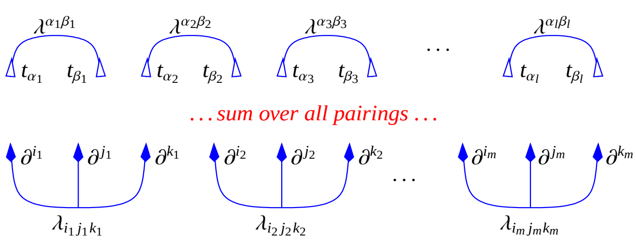
$$\int_{\mathbb{R}^n}e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}=\sum_{m\geq0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac12\lambda_{ij}x^ix^j}$$



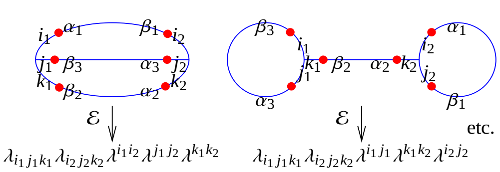
Feynman

$$=\sum_{m\geq0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0}=\sum_{\substack{m,l\geq0\\3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$=\sum_{\substack{m,l\geq0\\3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\Big[\ \Big]$$




$$=\sum_{\substack{m,l\geq0\\3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\\text{Feynman diagrams } D}}\mathcal{E}(D)$$




$$=C\sum_{\substack{\text{unmarked Feynman}\\\text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\text{Aut}(D)|}.$$

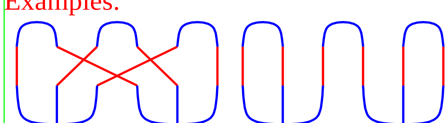
**Claim.** The number of pairings that produce a given unmarked Feynman diagram $D$ is $\frac{6^m m!2^l l!}{|\text{Aut}(D)|}$.

**Proof of the Claim.** The group $G_{m,l}:=[(S_3)^m\rtimes S_m]\times[(S_2)^l\rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings $P$ that produce a given $D$, and the stabilizer of any given $P$ is $\text{Aut}(D)$. □

**The Fourier Transform.** $(F\colon V\to\mathbb{C})\Rightarrow(\tilde{f}\colon V^*\to\mathbb{C})$ via $\tilde{F}(\varphi):=\int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0)=\int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f}\sim\widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})}\sim e^{Q^{-1}/2}$, where $Q$ is quadratic, $Q(v)=\langle Lv,v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi):=\langle\varphi,L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

**Examples.**



$|\text{Aut}(D)|=12$ $|\text{Aut}(D)|=8$

**Monsters left to Slay.**

- Convergence.
- Proof of invariance.
- The framing anomaly.
- Universallity.
- $d^{-1}$ doesn't really exist, Faddeev-Popov, determinants, ghosts, Berezin integration.
- Assembly.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506/