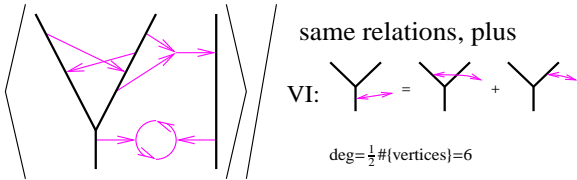
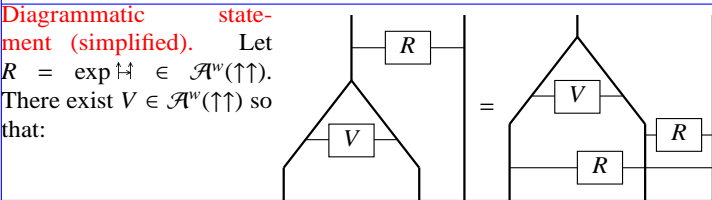


w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow \uparrow \uparrow)$ is



Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect R4.



Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(\mathfrak{g})^{\otimes 2}$ so that $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V e^{x+y} = e^x e^y V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$ -valued functions)

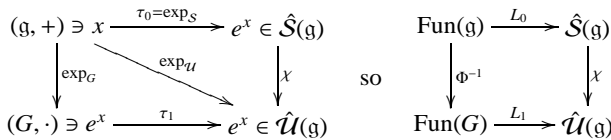
Group-Algebra statement (simplified). For every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)e^x e^y \quad (\text{shhh, this is Duflo})$$

Unitary \implies Group-Algebra. $\iint e^{x+y} \phi(x)\psi(y) = \langle 1, e^{x+y} \phi(x)\psi(y) \rangle = \langle V1, V e^{x+y} \phi(x)\psi(y) \rangle = \langle 1, e^x e^y V \phi(x)\psi(y) \rangle = \langle 1, e^x e^y \phi(x)\psi(y) \rangle = \iint e^x e^y \phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

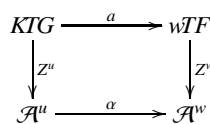
Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \rightarrow A$ is multiplicative then $(\text{Fun}(G), \star) \rightarrow (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie (G, \mathfrak{g}) ,



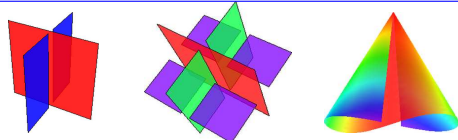
with $L_0 \psi = \int \psi(x) e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1 \Phi^{-1} \psi = \int \psi(x) e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

$u \leftrightarrow w$ The diagram on the right explains the relationship between associators and solutions of the Kashiwara-Vergne problem.



The Full
2-Knot Story



Question. Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

With $\kappa : (S = \mathbb{R}^2) \rightarrow M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

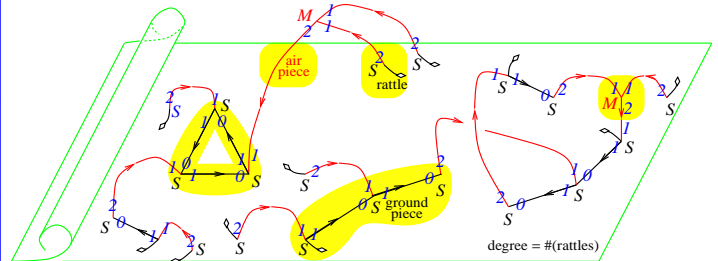
$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^* A} \alpha + \kappa^* B \rangle\right).$$

The BF Feynman Rules. For an edge e , let Φ_e be its direction, in S^3 or S^1 . Let ω_3 and ω_1 be volume forms on S^3 and S^1 . Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{[D]}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{e \in D} \Phi_e^* \omega_3 \prod_{e \in D} \Phi_e^* \omega_1$$

(modulo some IHX-like relations).

See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant on simple 2-knots.

- There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
- I don't know how to define / analyze “finite type” for general 2-knots.
- I don't know how to reduce Z_{BF} to combinatorics / algebra.

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.
 [BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ω/KBH , arXiv:1308.1721.
 [BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, $\omega/\text{WKO1}$, arXiv:1405.1956.
 [BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem*, $\omega/\text{WKO2}$, arXiv:1405.1955.
 [CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Math. Surv. and Mono. **55**, Amer. Math. Soc., Providence 1998.
 [CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.
 [KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions*, Invent. Math. **47** (1978) 249–272.
 [Ka] A. Kawauchi, *A Chord Diagram of a Ribbon Surface-Link*, <http://www.sci.osaka-cu.ac.jp/~kawauchi/>.
 [Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47–92, arXiv:math/0609742.



“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)

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