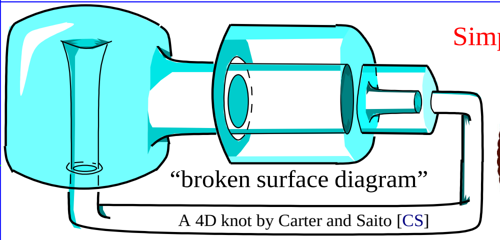
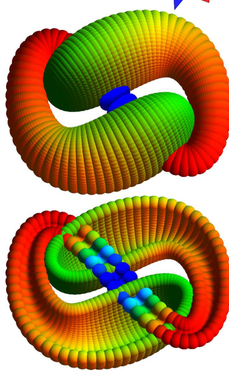
Simple 2-Knots.

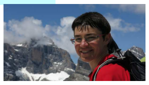


A 4D knot by Carter and Saito [CS]

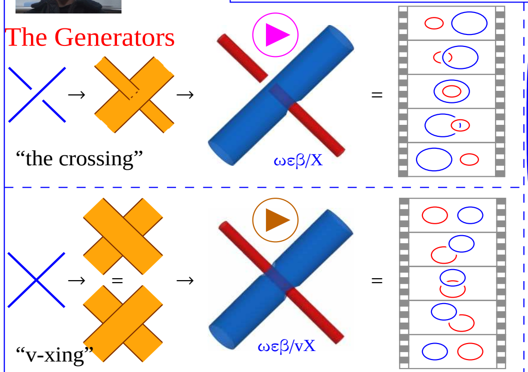


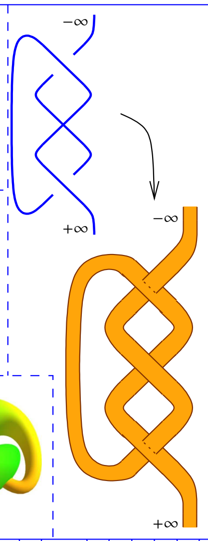
Satoh

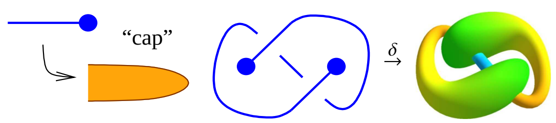
Dalvit ωεβ/Dal



The Generators

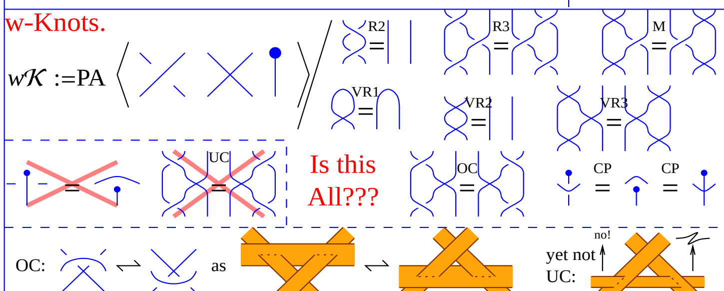






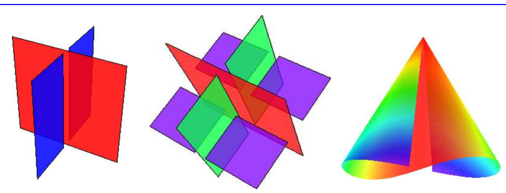
The Double Inflation Procedure δ.

w-Knots.



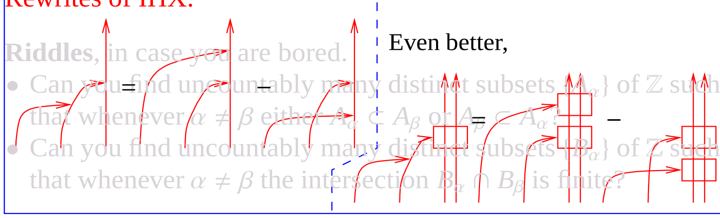
A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**.

The Full 2-Knot Story



Rewrites of IHX.

Even better,



Riddles, in case you are bored.

- Can you find uncountably many distinct subsets $\{A_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$?
- Can you find uncountably many distinct subsets $\{B_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ the intersection $B_\alpha \cap B_\beta$ is finite?

Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S,\mathfrak{g})$, $\alpha \in \Omega^1(S,\mathfrak{g}^*)$, set

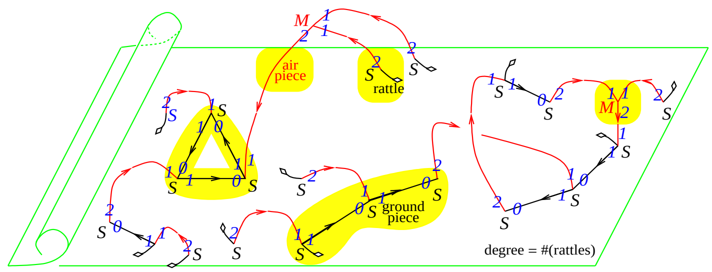
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$

Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some IHX-like relations). See also [Wa]



Issues.
- Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
- There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
- I don't know how to define / analyze "finite type" for general 2-knots.
- I don't know how to reduce Z_{BF} to combinatorics / algebra.

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamilton-1412/