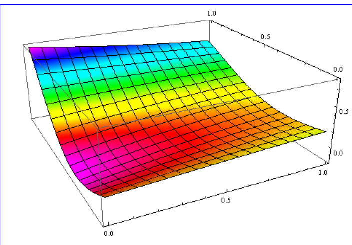
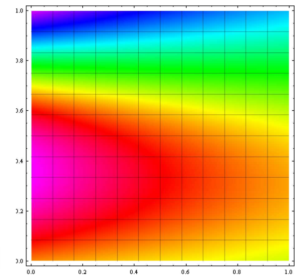
Dror Bar-Natan, Toronto November 2014. More at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411

Dessert: Hilbert's 13th Problem, in Full Colour

Abstract. To break a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnold solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.

In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnold showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function "+" (addition). For $f(x,y) = xy$, this may be $xy = \exp(\log x + \log y)$. For $f(x,y,z) = x^y/z$, this may be $\exp(\exp(\log y + \log\log x) + (-\log z))$. What might it be for (say) the real part of the Riemann zeta function?

The only original material in this talk will be the pictures; the math was known since around 1957.

$\frac{1}{3}\,\mathrm{Re}(\zeta(x+iy))$ on $[0,1]\times[13,17]$

Hilbert

Kolmogorov

Arnold (by Moser)

Fix an irrational $\lambda > 0$, say $\lambda = (\sqrt{5}-1)/2$. All functions are continuous.

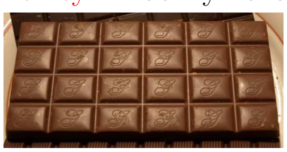
Theorem. There exist five $\phi_i : [0,1] \to [0,1]$ $(1 \le i \le 5)$ so that for every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that

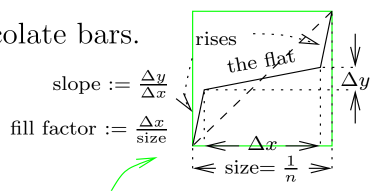
$$f(x,y) = \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))$$

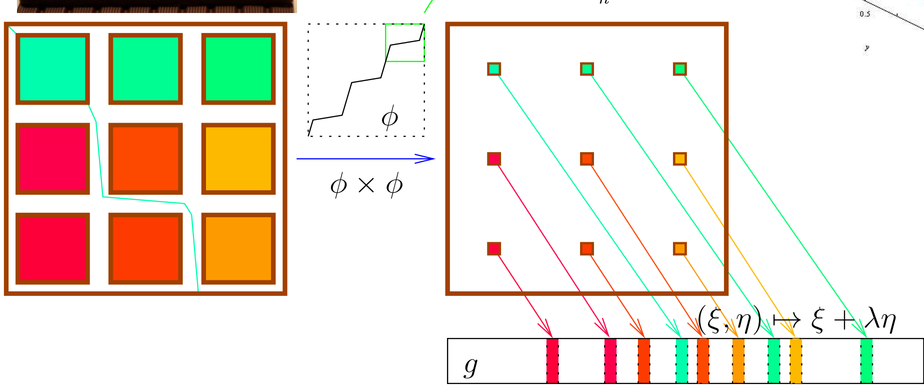
for every $x, y \in [0,1]$.

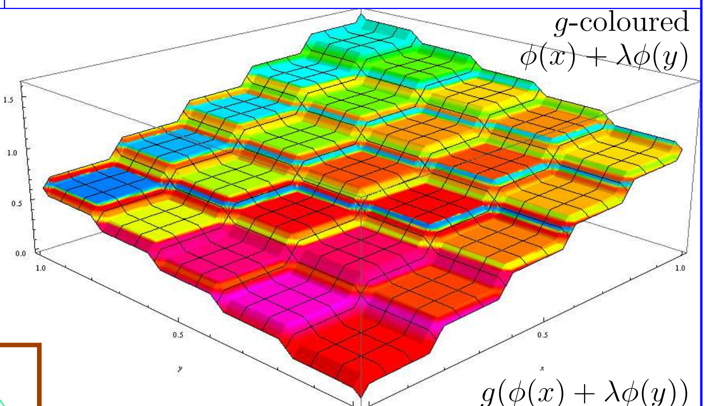
Step 1. If $\epsilon > 0$ and $f : [0,1]\times[0,1] \to \mathbb{R}$, then there exists $\phi : [0,1] \to [0,1]$ and $g : [0,1+\lambda] \to \mathbb{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on at least 98% of the area of $[0,1]\times[0,1]$.

The key. "Poorify" chocolate bars.

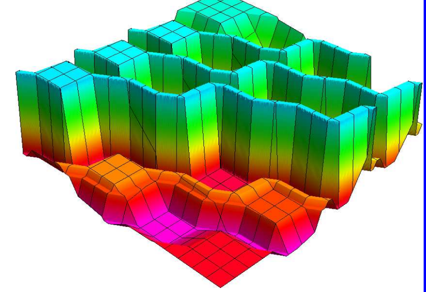




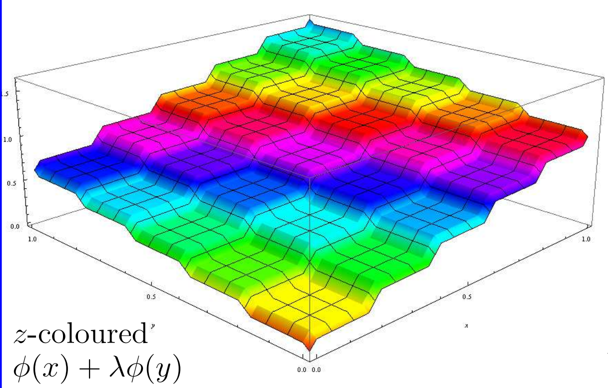




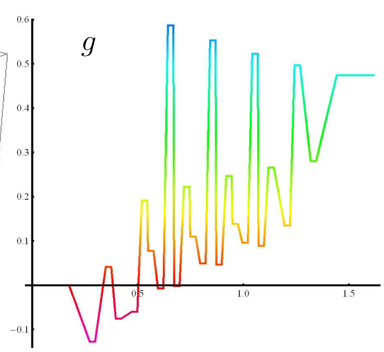
g-coloured $\phi(x) + \lambda\phi(y)$

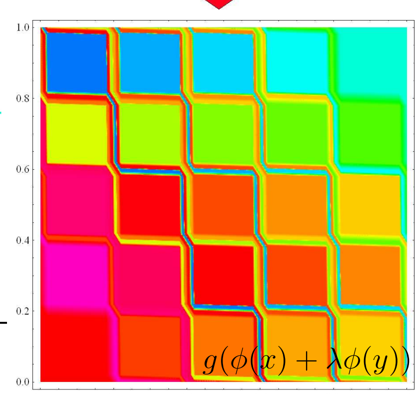


$g(\phi(x) + \lambda\phi(y))$



z-coloured' $\phi(x) + \lambda\phi(y)$





$g(\phi(x) + \lambda\phi(y))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/