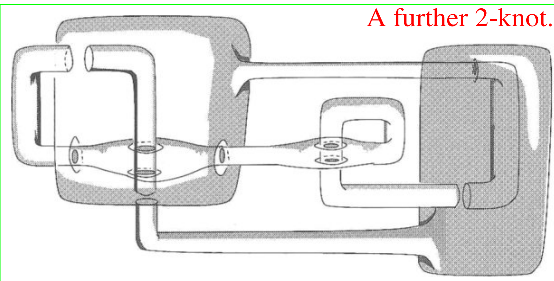
Knots in Three and Four Dimensions

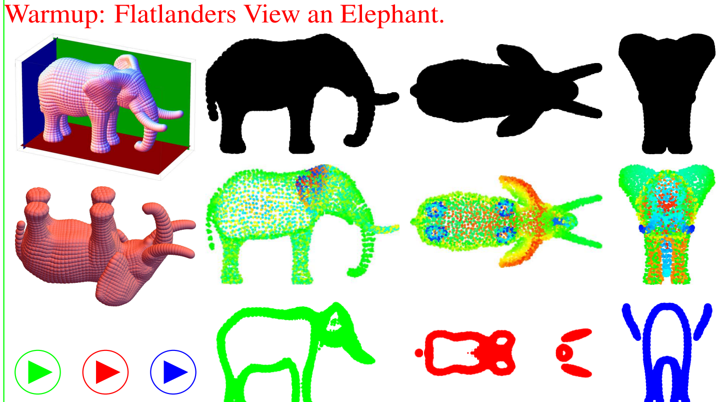
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

A further 2-knot.



ω/CS

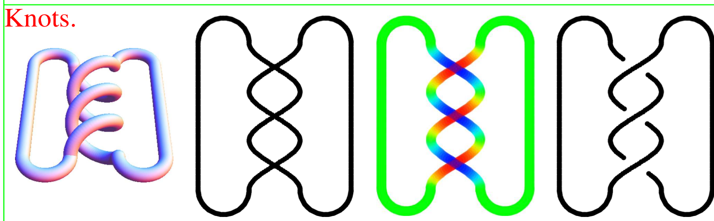
Warmup: Flatlanders View an Elephant.



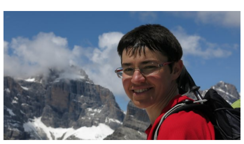
ω/g ω/r ω/b

coords from ω/Jeff2207

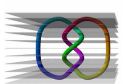
Knots.

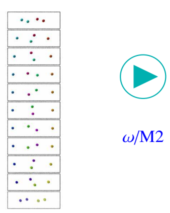


"broken curve diagram"



with Ester Dalvit ω/Dal





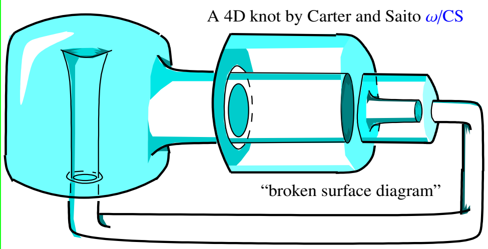
ω/M2

Formally, "a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such".

2-Knots / 4D Knots.

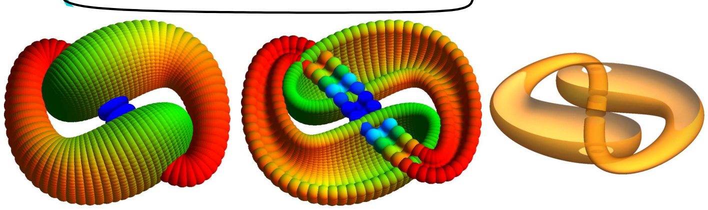
Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

A 4D knot by Carter and Saito ω/CS

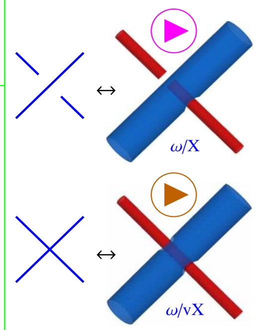


"broken surface diagram"

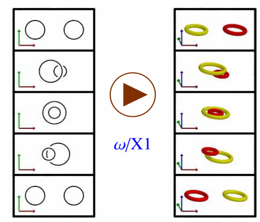
Carter, Banach, Saito



Some Movies

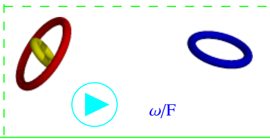


ω/X ω/vX

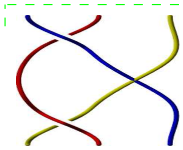


ω/X1

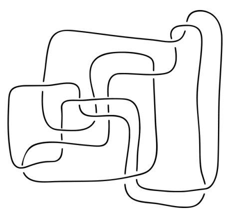
ω/Bub2



ω/F



Some Unknots



Thistlethwaite's unknot

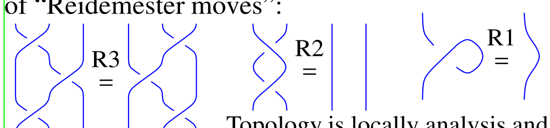
ω/U

Scharein's relaxation

Haken's unknot

Reidemeister's Theorem.

(a) Every knot has a "broken curve diagram", made only of curves and "crossings" like ╳. (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of "Reidemester moves":

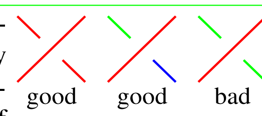


Kurt Reidemeister

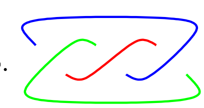
Topology is locally analysis and globally algebra

3-Colourings.

Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.

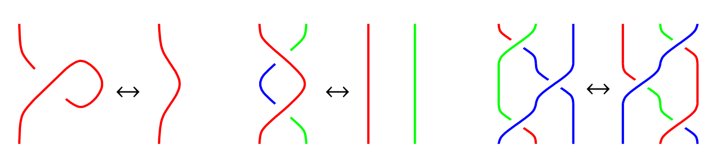


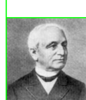
Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{trefoil}) = 9$; so $\bigcirc \neq$ trefoil.



Riddle. Is $\lambda(K)$ always a power of 3?

Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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