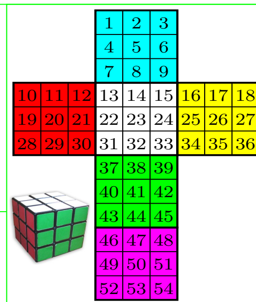
Non-Commutative Gaussian Elimination and Rubik's Cube

The Problem. Let $G = \langle g_1, \ldots, g_\alpha \rangle$ be a subgroup of S_n, with $n = O(100)$. Before you die, understand G:

1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \ldots, g_\alpha$.
4. Produce *random* elements of G.



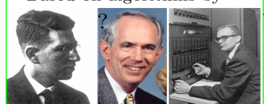
The Commutative Analog. Let $V = \text{span}(v_1, \ldots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	$\cdots$	$n-1$	n

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \ldots, *)$; $1 :=$ "the pivot".

Based on algorithms by



O. Schreier   C. Sims   D. Knuth

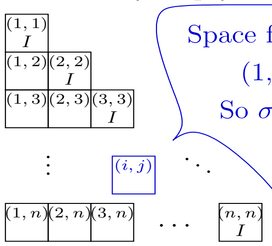
See also *Permutation Group Algorithms* by Á. Seress, *Efficient Representation of Perm Groups* by D. Knuth.

Feed $v_1, \ldots, v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.

1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,



Space for a $\sigma_{i,j} \in S_n$ of the form $(1, 2, \ldots, i-2, i-1, j, *, *, \ldots, *)$
So $\sigma_{i,j}$ fixes $1, \ldots, i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1, \ldots, g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j := \sigma(i)$.

1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim 1. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim 2. Every $\sigma_{i,j}$ in T is in G.

Claim 3. Anything fed in T is now a monotone product in T:
f was fed $\Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \;\&\; \sigma_{i,j_i} \in T\}$

The Generators

```
n = 54;
g1 = Cycles[{{1, 18, 45, 28}, {2, 27, 44, 19}, {3, 36, 43, 10}, {46, 52, 54, 48},
    {47, 49, 53, 51}}];
g2 = Cycles[{{7, 16, 39, 30}, {8, 25, 38, 21}, {9, 34, 37, 12}, {13, 15, 33, 31},
    {14, 24, 32, 22}}];
g3 = Cycles[{{28, 31, 34, 48}, {29, 32, 35, 47}, {30, 33, 36, 46}, {37, 39, 45, 43},
    {38, 42, 44, 40}}];
g4 = Cycles[{{1, 3, 9, 7}, {2, 6, 8, 4}, {10, 54, 16, 13}, {11, 53, 17, 14},
    {12, 52, 18, 15}}];
g5 = Cycles[{{1, 13, 37, 46}, {4, 22, 40, 49}, {7, 31, 43, 52}, {10, 12, 30, 28},
    {11, 21, 29, 19}}];
g6 = Cycles[{{3, 48, 39, 15}, {6, 51, 42, 24}, {9, 54, 45, 33}, {16, 18, 36, 34},
    {17, 27, 35, 25}}];
```
Enter

Claim 4. If two monotone products are equal,
$$\sigma_{1,j_1}\cdots\sigma_{n,j_n} = \sigma_{1,j'_1}\cdots\sigma_{n,j'_n},$$
then all the indices that appear in them are equal, $\forall i,\ j_i = j'_i$.

Claim 5. Let M_k denote the set of monotone products in T starting in column k:
$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$
then for every k, $M_k M_k \subset M_k$ (and so each M_k is a subgroup of G).

Proof. By backwards induction. Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:
$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \stackrel{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \stackrel{2}{\subset} M_4 M_5$$
$$\stackrel{3}{=} \cup_j \sigma_{4,j}(M_5 M_5) \stackrel{4}{\subset} \cup_j \sigma_{4,j} M_5 \subset M_4$$
(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case
$$(\sigma_{4,j'_4}\sigma_{5,j'_5}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$
falls like a chain of dominos.

Theorem. $G = M_1$ and we have achieved our goals.

A Demo Program

```
σ_ ∘ τ_  := PermutationProduct[τ, σ];
Feed[Cycles[{}]] := Null;
Feed[τ_] := Module[{i, j, k, l},
   i = Min[PermutationSupport[τ]];
   j = PermutationReplace[i, τ];
   If[Head[σi,j] === Cycles,
    Feed[InversePermutation[σi,j] ∘ τ],
    (*Else*) σi,j = τ;
    For[k = 1, k < n, ++k,
     For[l = k + 1, l ≤ n, ++l,
      If[Head[σk,l] === Cycles,
       Feed[σi,j ∘ σk,l]; Feed[σk,l ∘ σi,j]]
     ]]
   ]];
$RecursionLimit = ∞;
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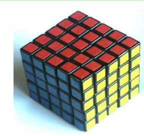

www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

Homework Problem 1. Can you do cosets?

www.powerstrike.net/puzzles/

The Results

```
Table[Feed[gα]; ∏ⁿi=1 (1 + Count[Range[n], j_ /; Head[σi,j] == Cycles]), {α, 6}]
{4, 16, 159 993 501 696 000, 21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}
```
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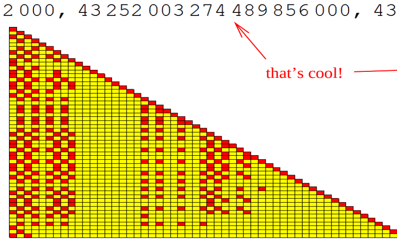
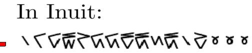


that's cool!

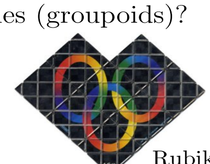
$$43{,}252{,}003{,}274{,}489{,}856{,}000 = \frac{8!\,3^8\,12!\,2^{12}}{12}$$

In Inuit:



Homework Problem 2. Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	



Rubik's magic

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Cambridge-1301/, see also http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/