Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar–Natan in Oxford, January 2013
ωεβ :=http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121

Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

$\mathcal{K}^{bh}(m,n)$.

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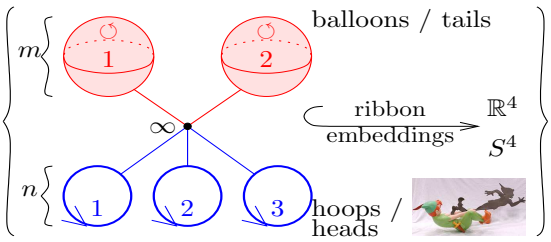
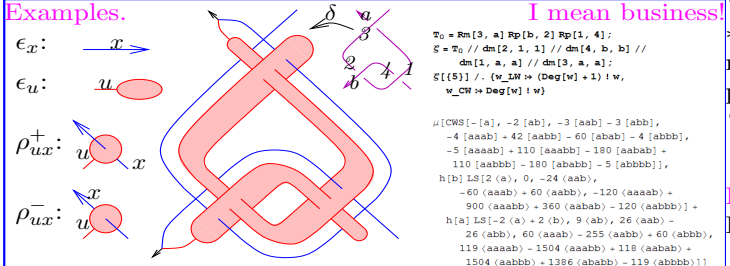
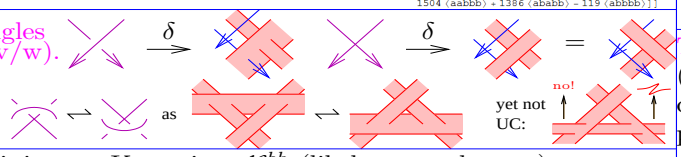


Examples. **I mean business!**

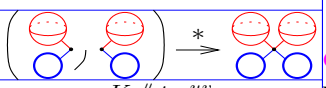


Tangles (u/v/w).



• δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
• δ maps v/w-tangles map to $\mathcal{K}^{bh}$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

Operations Punctures & Cuts; Connected Sums.

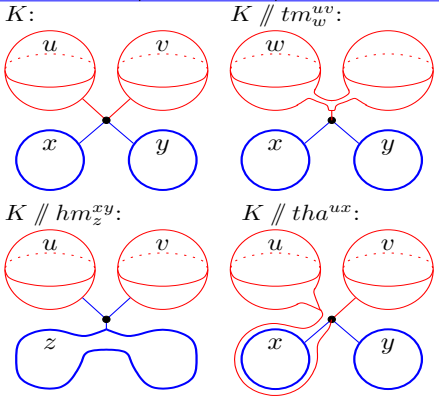


Meta-Group-Action.

If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

"MGA"

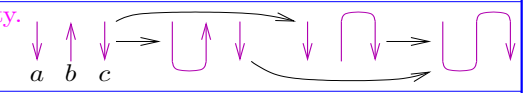
("$/\!/$" is newspeak for "apply an operator" and for "composition left to right")



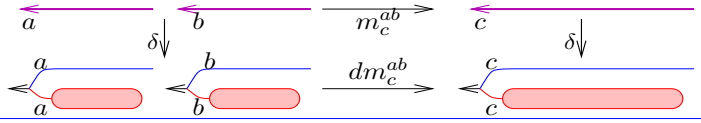
Properties.
• Associativities: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$, for $m = tm, hm$.
• Action axiom t: $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$,
• Action axiom h: $hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z$.
• SD Product: $dm^{ab}_c := tha^{ab} /\!/ tm^{ab}_c /\!/ hm^{ab}_c$ is associative.

Meta-associativity.

$m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$.

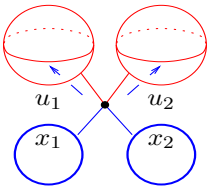


Tangle concatenations → $\pi_1 \ltimes \pi_2$.



Thus we seek homomorphic invariants of $\mathcal{K}^{bh}$!

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1. $*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longtitude":



Not computable! (but nearly)

$$(\Pi, (u,\dots), (x,\dots)) \mapsto (\Pi * \langle\bar{u}\rangle / (u = x\bar{u}x^{-1}), (\bar{u},\dots), (x,\dots))$$

Failure #0. Can we write the x's as free words in the u's? If $x = uv$, compute $x /\!/ tha^{ux}$:

$$x = uv \to \bar{u}v = u^x v = u^{\bar{u}v} v = u^{u^x v} v = u^{u^{u^x v} v} v = \cdots$$

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), and H a set of "head labels" ("hoop colours"). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T, so there's $\mathrm{tr} : FA \to CW$. Let $M(T,H) := \{(\bar{\lambda} = (x : \lambda_x)_{x\in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$

$$= \left\{ \left(x : \begin{array}{c} u \quad v \\ \diagdown\ \diagup \\ \bullet \end{array},\ y : \left|\ -\tfrac{22}{7} \begin{array}{c} u \quad u \\ \diagdown\ \diagup \\ \bullet \end{array} v \right.,\ \begin{array}{c} u \quad v \\ \diagdown\ \diagup \\ \bullet \\ \diagup\ \diagdown \\ v \quad v \end{array} \right) \dots \right\}$$

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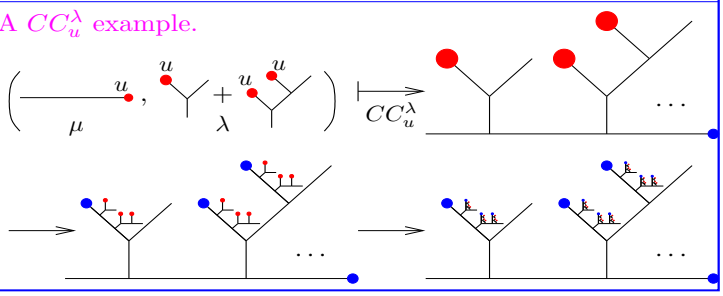
Operations. Set $(\bar{\lambda}_1;\omega_1) * (\bar{\lambda}_2;\omega_2) := (\bar{\lambda}_1 \cup \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda};\omega)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \Big(\big(\dots, \widehat{x : \lambda_x}, \widehat{y : \lambda_y}, \dots, z : \mathrm{bch}(\lambda_x, \lambda_y) \big) ; \omega \Big)$$

$$tha^{ux} : \mu \mapsto \underbrace{\mu \overbrace{/\!/\!/}^{\text{"stable apply"}} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar{u})) /\!/ (\bar{u} \mapsto u)}_{\mu /\!/ CC^{\lambda_x}_u} + \underbrace{(0; J_u(\lambda_x))}_{\text{the "}J\text{-spice"}}$$

A CC^λ_u example.



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