

# Local Khovanov Homology (1)

(an outdated overview)

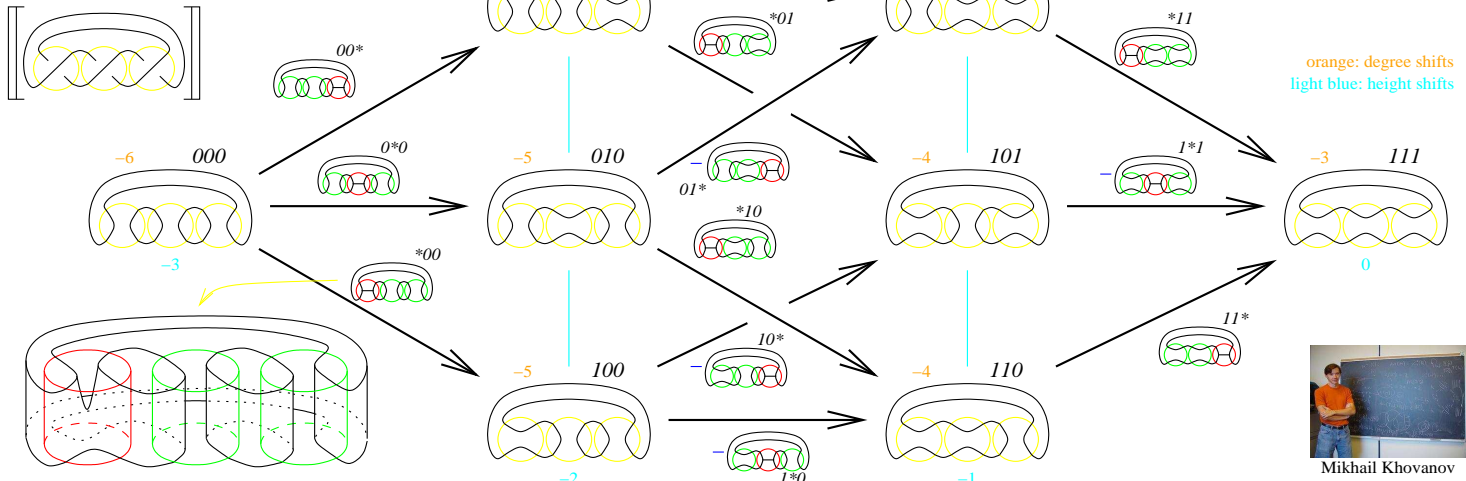
## The Jones polynomial:

$$J : \text{link} \mapsto q \langle -q^2 \text{crossing} \rangle, \quad J : \text{link} \mapsto -q^{-2} \langle \text{crossing} \rangle + q^{-1} \langle \text{crossing} \rangle$$

$$\bigcirc^k \mapsto (q + q^{-1})^k$$

$$J : \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle \mapsto -q^{-1} \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle + \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle + \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle - q \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle = -q^{-1} \langle \text{crossing} \rangle + (q + q^{-1}) \langle \text{crossing} \rangle - q \langle \text{crossing} \rangle$$

R2



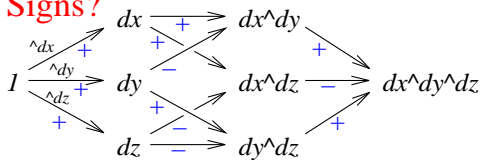
### What is it?

A cube for each knot/link projection;

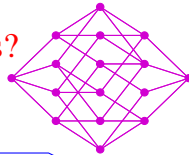
Vertices: All fillings of  $\bigcirc$  with  $\bigcirc$  or with  $\bigcirc$ .

Edges: All fillings of  $I \times \bigcirc$  with  $I \times \bigcirc$  or with  $I \times \bigcirc$  and precisely one  $\bigcirc$ .

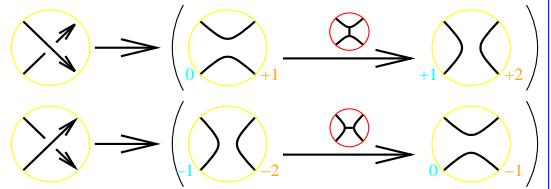
### Signs?



### More crossings?



### General Crossings



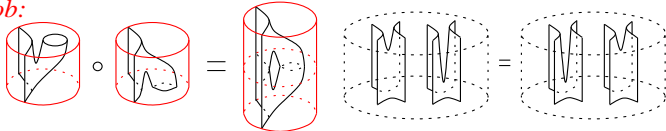
### Where does it live?

In  $Kom(Mat(\langle Cob \rangle / \{S, T, G, NC\})) / \text{homotopy}$

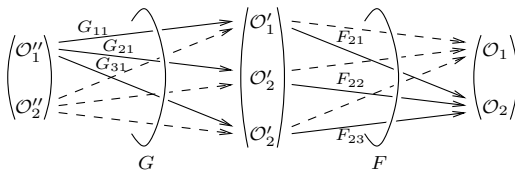
*Kom*: Complexes *Mat*: Matrices

*Cob*: Cobordisms  $\langle \dots \rangle$ : Formal lin. comb.

*Cob*:



*Mat(C)*:

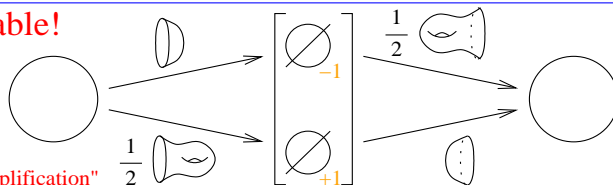


S:  $\bigcirc = 0$     T:  $\bigcirc = 2$     G:  $\bigcirc = 0$

NC:  $2 \bigcirc = \bigcirc + \bigcirc + \bigcirc$

### Computable!

via



"complex simplification"

### Complexes:

$$\Omega = (\Omega^{-n} \rightarrow \Omega^{-n+1} \rightarrow \dots \rightarrow \Omega^n)$$

### Morphisms:

$$\begin{array}{ccccccc} \dots & \rightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \rightarrow \dots \\ & & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & \\ \dots & \rightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \rightarrow \dots \end{array}$$

### Homotopies:

$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ \downarrow F^{r-1} & \swarrow h^r & \downarrow F^r & \swarrow h^{r+1} & \downarrow F^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

$$F^r - G^r = h^{r+1} d^r + d^{r-1} h^r$$

**The Main Point.** "The cube",  $Kh(L)$ , is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

- The Categorification Speculative Paradigm.**
- Every object in math is the Euler characteristic of a complex.
  - Every operation lifts to an operation between complexes.
  - Every identity remains true, up to homotopy.

All arrows in an arbitrary additive category.