Let x_c denote the path on which $\mathcal{L}(x)$ attains its Lemma 3.3. minimum value, write $x = x_c + x_q$ with $x_q \in W_{00}$, and get

$$\psi(T,x) = c \int dx_0 \psi_0(x_0) \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c + x_q)}.$$

In our particular case \mathcal{L} is quadratic in x, and therefore $\mathcal{L}(x_c + x_q) = \mathcal{L}(x_c) + \mathcal{L}(x_q)$ (this uses the fact that x_c is an extremal of \mathcal{L} , of course). Plugging this into what we already have, we get

$$\psi(T,x) = c \int dx_0 \psi_0(x_0) \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c) + i\mathcal{L}(x_q)}$$
$$= c \int dx_0 \psi_0(x_0) e^{i\mathcal{L}(x_c)} \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_q)}.$$

Now this is excellent news, because the remaining path integral over W_{00} does not depend on x_0 or x_n , and hence it is a constant! Allowing c to change its value from line to line, we get

$$\psi(T,x) = c \int dx_0 \psi_0(x_0) e^{i\mathcal{L}(x_c)}$$

Lemma 3.4 now shows us that $x_c(t) = x_0 \cos t +$ $x_n \sin t$. An easy explicit computation gives $\mathcal{L}(x_c) =$ $-x_0x_n$, and we arrive at our final result,

$$\psi(\frac{\pi}{2}, x) = c \int dx_0 \psi_0(x_0) e^{-ix_0 x_n}.$$

Notice that this is precisely the formula for the Fourier transform of ψ_0 ! That is, the answer to the question in the title of this document is "the particle gets Fourier transformed", whatever that may mean.

3. The Lemmas

Lemma 3.1. For any two matrices A and B,

$$e^{A+B} = \lim_{n \to \infty} \left(e^{A/n} e^{B/n} \right)^n.$$

Proof. (sketch) Using Taylor expansions, we see that $e^{\frac{A+B}{n}}$ and $e^{A/n}e^{B/n}$ differ by terms at most proportional to c/n^2 . Raising to the *n*th power, the two sides differ by at most O(1/n), and thus

$$e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{A+B}{n}} \right)^n = \lim_{n \to \infty} \left(e^{A/n} e^{B/n} \right)^n,$$

as required.

Lemma 3.2.

$$\left(e^{itV}\psi_0\right)(x) = e^{itV(x)}\psi_0(x)$$

$$\left(e^{i\frac{t}{2}\Delta}\psi_0\right)(x) = c\int dx' e^{i\frac{(x-x')^2}{2t}}\psi_0(x')$$

Proof. In fact, the left hand side of this equality is just a solution $\psi(t, x)$ of Schrödinger's equation with V = 0:

$$\frac{\partial \psi}{\partial t} = \frac{i}{2} \Delta_x \psi, \qquad \psi|_{t=0} = \psi_0.$$

Taking the Fourier transform $\psi(t,p)$ _ $\frac{1}{\sqrt{2\pi}}\int e^{-ipx}\psi(t,x)dx$, we get the equation

$$\frac{\partial \tilde{\psi}}{\partial t} = -i \frac{p^2}{2} \tilde{\psi}, \qquad \tilde{\psi}|_{t=0} = \tilde{\psi}_0.$$

For a fixed p, this is a simple first order linear differential equation with respect to t, and thus,

$$\tilde{\psi}(t,p) = e^{-i\frac{tp^2}{2}}\tilde{\psi}_0(p)$$

Taking the inverse Fourier transform, which takes products to convolutions and Gaussians to other Gaussians, we get what we wanted to prove.

Lemma 3.4. With the notation of Section 2 and at the specific case of $V(x) = \frac{1}{2}x^2$ and $T = \frac{\pi}{2}$, we have

$$x_c(t) = x_0 \cos t + x_n \sin t.$$

Proof. If x_c is a critical point of \mathcal{L} on $W_{x_0x_n}$, then for any $x_q \in W_{00}$ there should be no term in $\mathcal{L}(x_c + \epsilon x_q)$ which is linear in ϵ . Now recall that

$$\mathcal{L}(x) = \int_0^T dt \left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right),$$

so using $V(x_c + \epsilon x_q) \sim V(x_c) + \epsilon x_q V'(x_c)$ we find that the linear term in ϵ in $\mathcal{L}(x_c + \epsilon x_q)$ is

$$\int_0^T dt \left(\dot{x}_c \dot{x}_q - V'(x_c) x_q \right)$$

Integrating by parts and using $x_q(0) = x_q(T) = 0$, this becomes

$$\int_0^T dt \left(-\ddot{x}_c - V'(x_c) \right) x_q.$$

For this integral to vanish independently of x_a , we must have $-\ddot{x}_c - V'(x_c) \equiv 0$, or

$$= -V'(x_c).$$
 (This is the famous $F = ma'$
of Newton's, and we have just
rediscovered the principle of
least action!

In our particular case this boils down to the equation

$$\ddot{x}_c = -x_c, \qquad x_c(0) = x_0, \qquad x_c(\pi/2) = x_n,$$

whose unique solution is displayed in the statement \Box of this lemma.

Video and more at http://drorbn.net/?title=AKT-14 (Jan 10 and Jan 17 classes)

 \ddot{x}_c