Lecture 2 Handout

More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \longmapsto A/\sqrt{k}$, and setting $\hbar = \frac{1}{\sqrt{k}}$:

$$Z(\gamma) = \int_{A \in \Omega^1(\mathbb{R}^3, \mathfrak{g})} \mathcal{D}A \, \operatorname{tr}_R \operatorname{hol}_\gamma(A) \, e^{\underbrace{\frac{i}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}(A \wedge dA + \frac{2\hbar}{3} A\wedge A \wedge A)}_{CS(A)}}$$

where $\operatorname{tr}_R \operatorname{hol}_\gamma(A) = \operatorname{tr}_R\Big(1 + \hbar \int ds\, A(\dot\gamma(s))$

$$+ \hbar^2 \int_{s_1 < s_2} A(\dot\gamma(s_1)) A(\dot\gamma(s_2)) + \dots$$

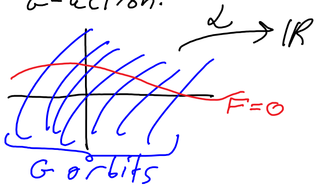
Trouble! "d" is not invertible!

Gauge Invariance: $CS(A)$ is invariant under $A \mapsto A + \delta A$, $\delta A = -(dC + \hbar[A, C])$, $C \in \Omega^0(\mathbb{R}, \mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional Group G w/ Lie algebra $\mathfrak{g} = \langle g_a \rangle$, and suppose $F: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is such that $F=0$ is a section of the G-action:

$$G \longrightarrow \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$



Then

$$\int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \delta(F(x)) \cdot \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^l} d\phi \, e^{i(\mathcal{L} + F(x)\cdot\phi)} \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

Perturbation theory for determinants?

$$\det(J_0 + \hbar J_1(x)) = \det(J_0) \sum_m \hbar^m \operatorname{Tr}(\Lambda^m J_0^{-1}) \cdot (\Lambda^m J_1(x))$$

Berezin
Fermionic
Anti-commuting
Variables: $\int d^k\bar{c}\, d^k c\, e^{i\bar{c}_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar{c} \int d^k c \, e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar{c}_a \left(\frac{\partial F^a}{\partial g_b}\right) c^b}_{\text{“ghosts”}}$$

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

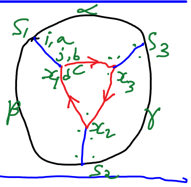
$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\big(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A + \phi\, \partial_i A^i$$
$$+ \bar{c}\, \partial_i (\partial^i + \operatorname{ad} A^i) c$$

So we have

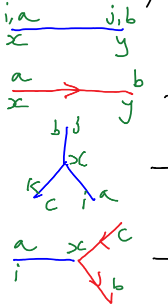
- * A bosonic quadratic term involving $\binom{A}{\phi}$.
- * A fermionic quadratic term involving $\bar{c}, c$.
- * A cubic interaction of 3 A's.
- * A cubic $A\bar{c}c$ vertex.
- * Funny A and γ "holonomy" vertices along γ.

After much crunching:

$$Z(\gamma) = \sum_{m=0}^{\infty} \hbar^m \sum_{\text{Feynman diags } D} \oint \mathcal{E}(D)$$

$D =$ 

where $\mathcal{E}(D)$ is constructed as follows:

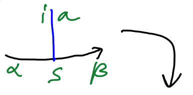


$$\longrightarrow \epsilon_{ijk} t^{ab} \frac{i(x-y)^k}{2|x-y|^3}$$

$$\longrightarrow \frac{t^{ab}}{4\pi|x-y|}$$

$$\longrightarrow \frac{i}{2\pi}\int_{\mathbb{R}^3} t_{abc}\, \epsilon^{ijk}$$

$$\longrightarrow \frac{1}{2\pi}\int_{\mathbb{R}^3} dz\, t_{abc}\, \underbrace{\partial_x^i}_{\text{acting only in } b\text{-direction}}$$



$$\int_{S^1} ds\, R^\alpha_{a\beta}\, \dot\gamma^i(s)$$

(−) sign for each red loop.

By a bit of a miracle, this boils down to a configuration space integral, which in itself, can be reduced to a pre-image count.

... But I run out of steam for tonight...

Banks like knots.