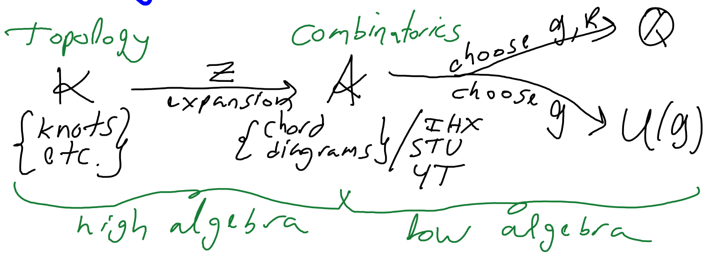
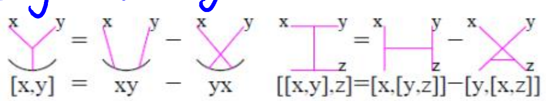
Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.



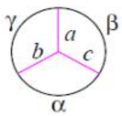
Very low algebra.



More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}$:  $\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = gl_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

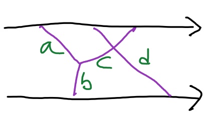
Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal{U}(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $\mathcal{U}(\mathfrak{g})$.
* $\exists\, \Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y \rangle / [x,y] = x$, determine $\mathcal{U}(\mathfrak{g})$. Guess a generalization.

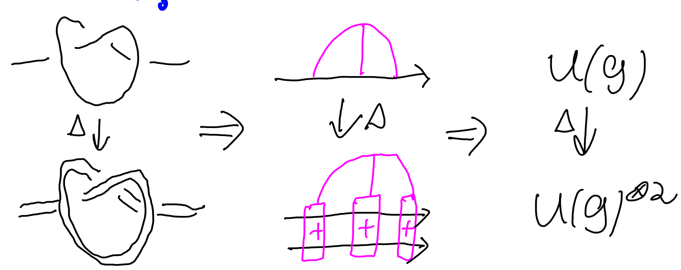
Low algebra. $\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via



$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$

& likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n}$ $\Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

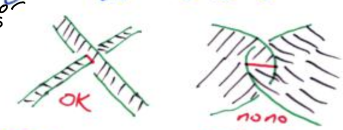


A "Homomorphic Expansion" $Z : \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra**.

$\{\text{Ribbon knots}\} \overset{\text{more or less}}{=} \{u\gamma : \gamma \in \mathcal{K}(\text{O-O}),\ d\gamma = \text{OO}\}$

where Ribbon means



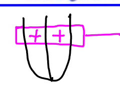
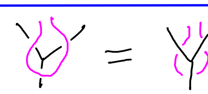
and

Algebraic knot theory:

$$\begin{tikzcd} & \mathcal{K}(\text{OO}) \arrow[r, "Z"] & \mathcal{A}(\text{OO}) \ni \text{OO} \\ \mathcal{K}(\text{O-O}) \arrow[ur, "d"] \arrow[dr, "u"] \arrow[rr, "Z"] & & \mathcal{A}(\text{O-O}) \arrow[u, "d"] \arrow[d, "u"] \\ & \mathcal{K}(\text{O}) \arrow[r, "Z"] & \mathcal{A}(\text{O}) \end{tikzcd}$$

So

$Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(\text{OO})\} \subset \mathcal{A}(\text{O-O})$

VI  $= 0$, follows from 

An Associator:

Quantum Algebra's "root object"

$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$

satisfying the "pentagon",

$$\begin{tikzcd} ((AB)C)D \arrow[r] \arrow[d, "\Phi 1"] & (AB)(CD) \arrow[r, phantom, ""] & \\ (A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & & \\ & A((BC)D) \arrow[r, "1\Phi"'] & A(B(CD)) \end{tikzcd}$$

with $((AB)C)D \xrightarrow{(\Delta 11)\Phi} (AB)(CD) \xrightarrow{(11\Delta)\Phi} A(B(CD))$.

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/