Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

This is an overview with too many and not enough details. I apologize.

Dror Bar−Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

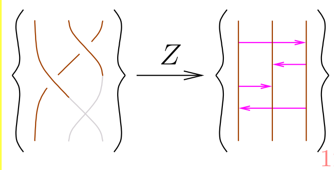
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle\ker\mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.

Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filterred $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n =$ "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



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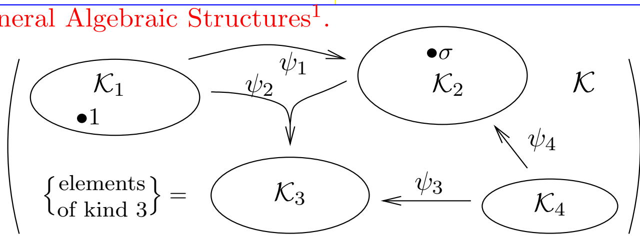
General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations. 14

All still works!

Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Leibniz[2] algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

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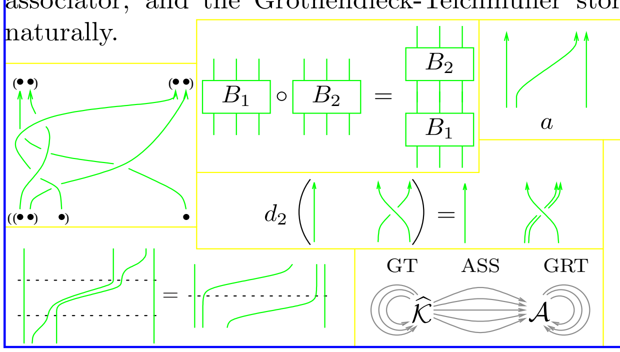
Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[3] arises naturally.



Drinfel'd [Dr1,2]

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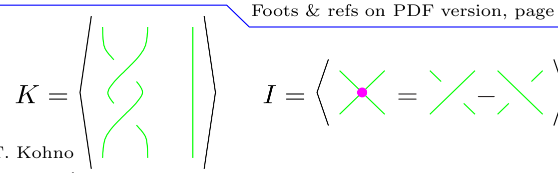
Example 1.

T. Kohno

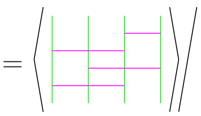


$$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1}$ $\quad C = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \,|\ \,|\!\!-\!\!|\, \rangle$

$$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle \text{4T relations} \rangle$$

$\mathcal{A} = A_n = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} =$

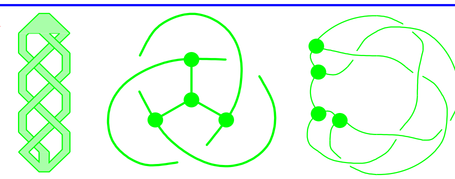


Z: universal finite type invariant, the Kontsevich integral. 9

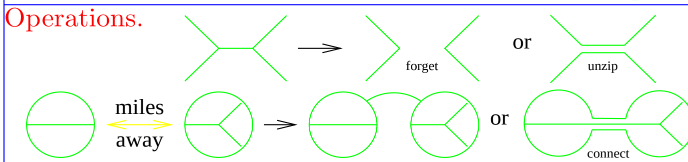
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

Example 5 - Knotted Trivalent Graphs.

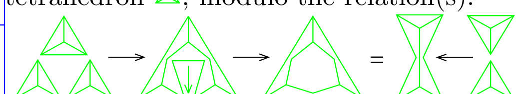
D. Thurston [Th]



Operations.

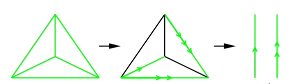


Presentation. KTG is generated by ribbon twists and the tetrahedron $\triangle$, modulo the relation(s):



(+more)

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.



$:= \Phi \in \mathcal{A}(\uparrow_3)$

A $\mathcal{U}(\mathfrak{g})$-Associator:

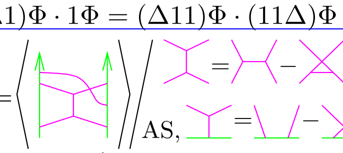
$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{array}{ccc} ((AB)C)D & \xrightarrow{(\Delta 11)\Phi} & (AB)(CD) \\ \downarrow{\scriptstyle \Phi 1} & & \downarrow{\scriptstyle (11\Delta)\Phi} \\ (A(BC))D & & A(B(CD)) \\ & {\scriptstyle (1\Delta 1)\Phi} \searrow \quad A((BC)D) \quad \nearrow {\scriptstyle 1\Phi} & \end{array}$$

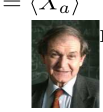
$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

$\mathcal{A}(\uparrow_2) :=$

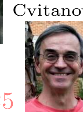


(deg $= \frac{1}{2}\#\{$trivalent vertices$\}$)

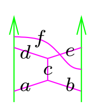
$$\xrightarrow[\text{metrized } \mathfrak{g}\ \ \mathfrak{g} = \langle X_a \rangle]{\text{Given a}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$$



Penrose



Cvitanovic



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/