

1. $\text{proj } \mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \rtimes \mathfrak{tr}_n)$

— All Signs Are Wrong! —

Dror Bar-Natan, Montpellier, June 2010, <http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/>

I understand Drinfel'd and Alekseev-Torossian, I don't understand Etingof-Kazhdan yet, and I'm clueless about Kontsevich

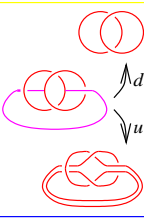
Cans and Can't Yet.

(arbitrary algebraic structure) $\xrightarrow[\text{machine}]{\text{projectivization}}$ (a problem in graded algebra)

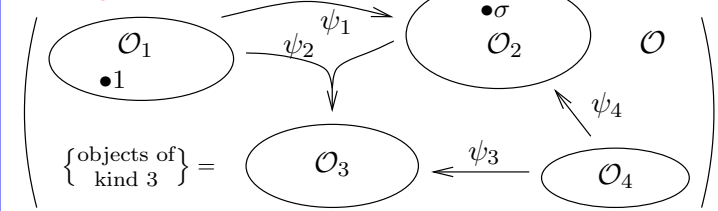
- Feed knot-things, get Lie algebra things.
- (u-knots) \rightarrow (Drinfel'd associators).
- (w-knots) \rightarrow (K-V-A-E-T).
- Dream: (v-knots) \rightarrow (Etingof-Kazhdan).
- Clueless: (???) \rightarrow (Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.



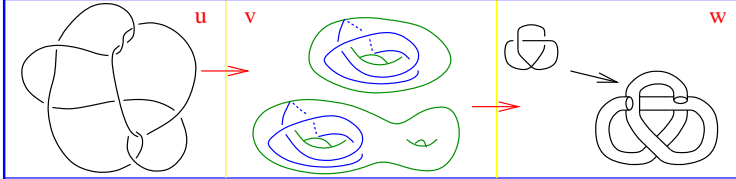
www.katlas.org



"An Algebraic Structure"



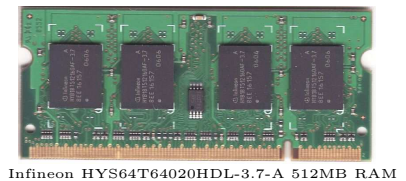
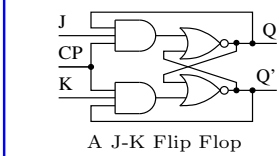
- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.



u-Knots (PA := Planar Algebra)

{knots & links} = PA $\langle \diagdown, \diagup, \text{R123: } \begin{matrix} \diagdown \\ \diagup \end{matrix} \rangle_{0 \text{ legs}}$

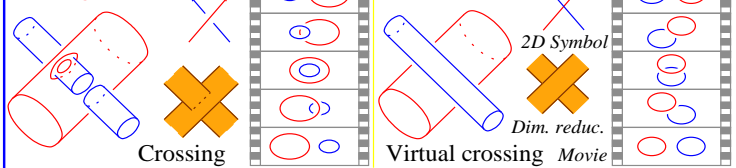
Circuit Algebras



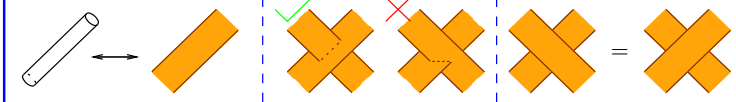
v-Tangles and w-Tangles (CA := Circuit Algebra)

{v-knots & links} = CA $\langle \diagdown, \diagup, \text{R23: } \begin{matrix} \diagdown \\ \diagup \end{matrix} \rangle$
 = PA $\langle \diagdown, \diagup, \text{VR123: } \begin{matrix} \diagdown \\ \diagup \end{matrix}, \text{R23: } \begin{matrix} \diagdown \\ \diagup \end{matrix} \rangle$
 {w-Tangles} = v-Tangles / OC: $\begin{matrix} \diagdown \\ \diagup \end{matrix} = \begin{matrix} \diagup \\ \diagdown \end{matrix}$

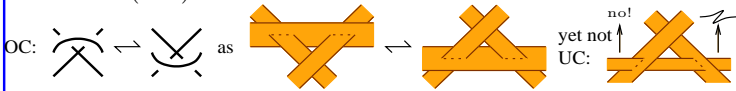
The w-generators.



A Ribbon 2-Knot is a surface S embedded in \mathbb{R}^4 that bounds an immersed handlebody B , with only "ribbon singularities"; a ribbon singularity is a disk D of transverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



"God created the knots, all else in topology is the work of mortals."
 Leopold Kronecker (modified)
 Also see <http://www.math.toronto.edu/~drorbn/papers/WKO/>

Homomorphic expansions for a filtered algebraic structure \mathcal{K} :

$$\text{ops } \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

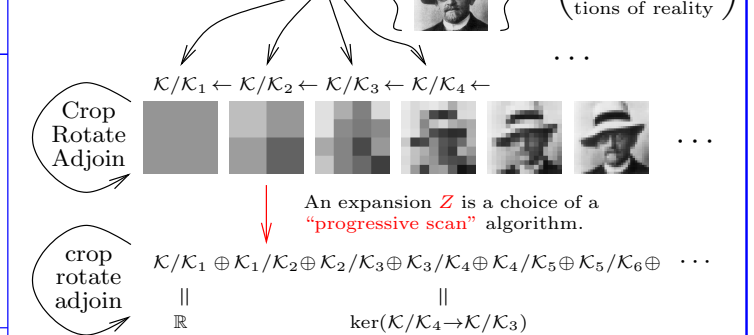
$$\downarrow \qquad \qquad \qquad \downarrow Z$$

$$\text{ops } \mathcal{K} \text{ gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtered $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$ that "covers" the identity on $\text{gr } \mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Reality. $\text{gr } \mathcal{K}$ is often too hard. An \mathcal{A} -expansion is a graded "guess" \mathcal{A} with a surjection $\tau : \mathcal{A} \rightarrow \text{gr } \mathcal{K}$ and a filtered $Z : \mathcal{K} \rightarrow \mathcal{A}$ for which $(\text{gr } Z) \circ \tau = I_{\mathcal{A}}$. An \mathcal{A} -expansion confirms \mathcal{A} and yields an ordinary expansion. Same for "homomorphic".

Just for fun.



Filtered algebraic structures are cheap and plenty. In any \mathcal{K} , allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products"). In this case, set $\text{proj } \mathcal{K} := \text{gr } \mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

2. Pure braids — PB_n is generated by x_{ij} , "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \text{gr } PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the 4T relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n , including the Drinfel'd theory of associators.

3. Quandle: a set Q with an op \wedge s.t.
 $1 \wedge x = 1, \quad x \wedge 1 = x, \quad (\text{appetizers})$
 $(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad (\text{main})$

$\text{proj } Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate $I!$), feed $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$



Kashiwara, Vergne, Alekseev, Enriquez, Torossian.