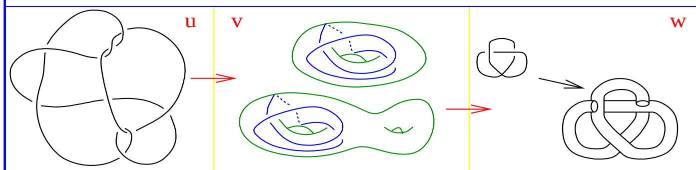
Dror Bar−Natan, Goettingen, April 2010


## Plans and Dreams

$$\begin{pmatrix}\text{arbitrary algebraic}\\ \text{structure}\end{pmatrix} \xrightarrow[\text{machine}]{\text{projectivization}} \begin{pmatrix}\text{a problem in}\\ \text{graded algebra}\end{pmatrix}$$

- Feed knot-things, get Lie algebra things.
- Feed u-knots, get Drinfel'd associators.
- Feed w-knots, get Kashiware-Vergne-Alekseev-Torossian.
- Dream: Feed v-knots, get Etingof-Kazhdan.
- Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.

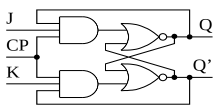



## u-Knots

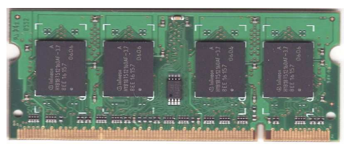
(PA :=Planar Algebra)

$$\begin{Bmatrix}\text{knots}\\ \text{\&links}\end{Bmatrix} = \text{PA}\left\langle \ \middle|\ \text{R123:}\ \right\rangle_{0 \text{ legs}}$$

## Circuit Algebras



A J-K Flip Flop


Infineon HYS64T64020HDL-3.7-A 512MB RAM

## v-Knots

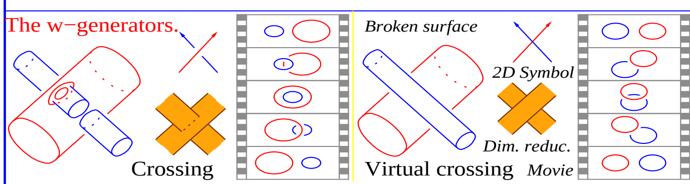
(CA :=Circuit Algebra)

$$\begin{Bmatrix}\text{v-knots}\\ \text{\& links}\end{Bmatrix} = \text{CA}\left\langle \ \middle|\ \text{R23:}\ \right\rangle_{0 \text{ legs}}$$

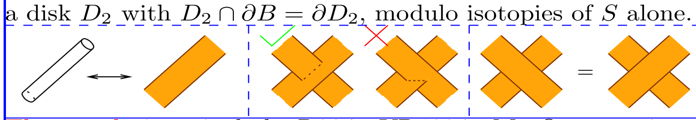
$$= \text{PA}\left\langle \ \middle|\ \text{VR123:}\ ;\quad \text{R23; D:}\ \right\rangle_{0 \text{ legs}}$$

## w-Tangles

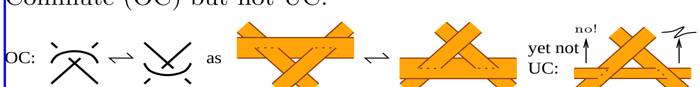
$$\{\text{w-Tangles}\} = \text{v-Tangles} \Big/ OC :$$




**A Ribbon 2-Knot** is a surface $S$ embedded in $\mathbb{R}^4$ that bounds an immersed handlebody $B$, with only "ribbon singularities"; a ribbon singularity is a disk $D$ of trasverse double points, whose preimages in $B$ are a disk $D_1$ in the interior of $B$ and a disk $D_2$ with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of $S$ alone.

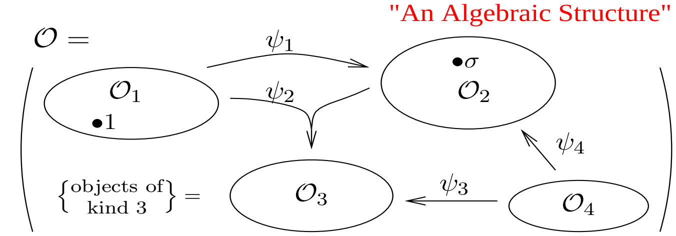


**The w-relations** include R234, VR1234, M, Overcrossings Commute (OC) but not UC:




Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

## "An Algebraic Structure"




- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

**Homomorphic expansions** for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\Downarrow \qquad \downarrow Z$$

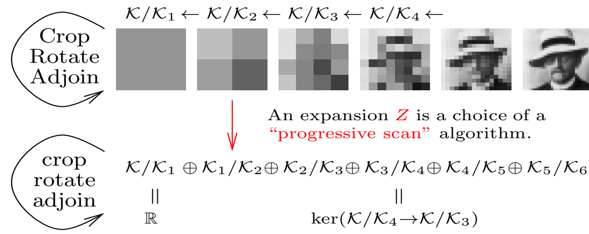
$$\text{ops} \circlearrowright \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

**Just for fun.**

$$\mathcal{K} = \left\{ \ \right\} = \begin{pmatrix}\text{The set of all}\\ \text{b/w 2D projec-}\\ \text{tions of reality}\end{pmatrix}$$

$$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \quad \dots$$




An expansion $Z$ is a choice of a "progressive scan" algorithm.

crop rotate adjoin $\quad \mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \quad \cdots$

$$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}, \qquad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$$

**Filtered algebraic structures are cheap and plenty.** In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

**Examples.** 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set $Q$ with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

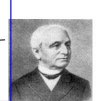
$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate $I$!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

**Our case(s).**

$$\mathcal{K} \xrightarrow[\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{proj}\mathcal{K} \xrightarrow[\substack{\text{low algebra: pic-}\\ \text{tures represent}\\ \text{formulas}}]{\substack{\text{given a "Lie"}\\ \text{algebra } \mathfrak{g}}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{proj}\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/