

Day 2 – u, v, w: combinatorics, low and high algebra

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The Scheme. Topology → Combinatorics → Lie Theory via

$$\mathcal{K} \xrightarrow[\text{equations, unknowns}]{Z: \text{high algebra}} \mathcal{A} = \text{proj } \mathcal{K} = \bigoplus \mathcal{I}^m / \mathcal{I}^{m+1} \xrightarrow[\text{pictures} \rightarrow \text{formulas}]{\mathcal{T}_g: \text{low algebra}} \mathcal{U}(\mathfrak{g})$$

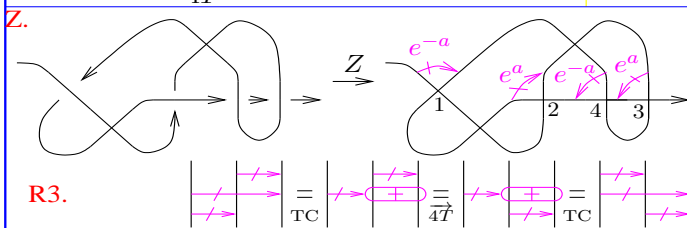
$1+1=2$, on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \cong \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (with $\bigcirc - \# - \bigcirc = -\bigcirc$)
 T. Le and D. Thurston.

The Finite Type Story. With $\bowtie := \times - \times$ set $\mathcal{V}_m := \{V : wK \rightarrow \mathbb{Q} : V(\bowtie^{>m}) = 0\}$. $\bigoplus (\mathcal{V}_m / \mathcal{V}_{m-1})^*$

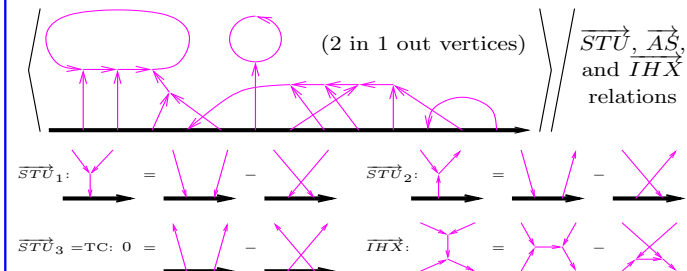
$$\mathcal{R} = \langle \frac{\text{TC}}{4T} \rangle \rightarrow \mathcal{D} = \langle \text{m arrows} \rangle \xrightarrow{\pi} \bigoplus (\mathcal{V}^m / \mathcal{V}^{m+1}) \rightarrow 0$$

$$\mathcal{A}^w := \mathcal{D} / \mathcal{R} \xleftarrow{\text{(filtered)}} wK \xrightarrow{Z} \text{gr } Z \xrightarrow{(\text{gr } Z) \circ \pi = I} \mathcal{A}^w$$

I take pride in this box



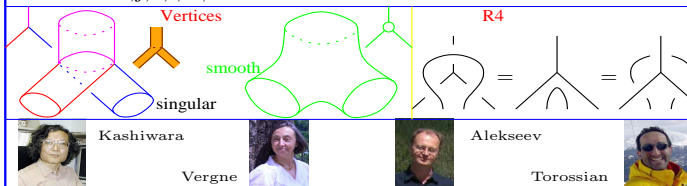
The Bracket-Rise Theorem. \mathcal{A}^w is isomorphic to



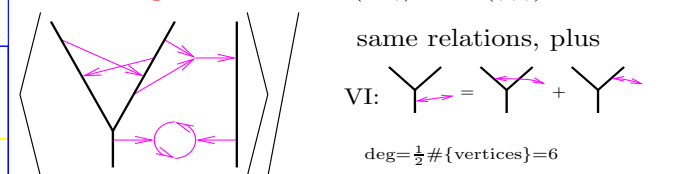
Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Low Algebra. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via

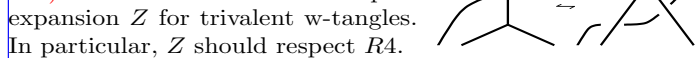
$$\sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^n \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$



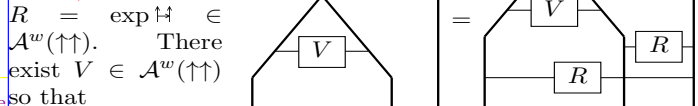
w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow \uparrow \uparrow)$ is



Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect R4.



Diagrammatic statement (simplified). Let $R = \exp \mathbb{H} \in \mathcal{A}^w(\uparrow \uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow \uparrow)$ so that



Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \mathbb{Q}$ so that $V(\Delta \otimes 1)(R) = R^{13} R^{23} V$ in $\hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V e^{x+y} = \hat{e}^x \hat{e}^y V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$ -valued functions)

Unitary \iff Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$: $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad } x$: $(x\varphi)(y) := \varphi([x, y])$.

Group-Algebra statement (simplified). For every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x) \psi(y) e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x) \psi(y) e^x e^y \quad (\text{shhh, this is Duflo})$$

Unitary \implies Group-Algebra. $\iint e^{x+y} \phi(x) \psi(y) = \langle 1, e^{x+y} \phi(x) \psi(y) \rangle = \langle V1, V e^{x+y} \phi(x) \psi(y) \rangle = \langle 1, e^x e^y V \phi(x) \psi(y) \rangle = \langle 1, e^x e^y \phi(x) \psi(y) \rangle = \iint e^x e^y \phi(x) \psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \rightarrow A$ is multiplicative then $(\text{Fun}(G), \star) \rightarrow (A, \cdot)$ via $L : f \mapsto \sum f(a) \tau(a)$. For Lie (G, \mathfrak{g}) ,

$$\begin{array}{ccc} (\mathfrak{g}, +) \ni x \xrightarrow{\tau_0 = \exp_S} e^x \in \hat{\mathcal{S}}(\mathfrak{g}) & & \text{Fun}(\mathfrak{g}) \xrightarrow{L_0} \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_U & \downarrow \chi \\ (G, \cdot) \ni e^x \xrightarrow{\tau_1} e^x \in \hat{\mathcal{U}}(\mathfrak{g}) & & \text{Fun}(G) \xrightarrow{L_1} \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} & & \downarrow \Phi^{-1} \\ & & \downarrow \chi \end{array}$$

with $L_0 \psi = \int \psi(x) e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1 \Phi^{-1} \psi = \int \psi(x) e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$$\star \text{ in } G : \iint \psi_1(x) \psi_2(y) e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x) \psi_2(y) e^{x+y}$$