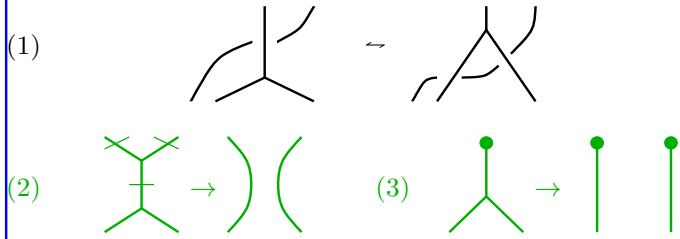
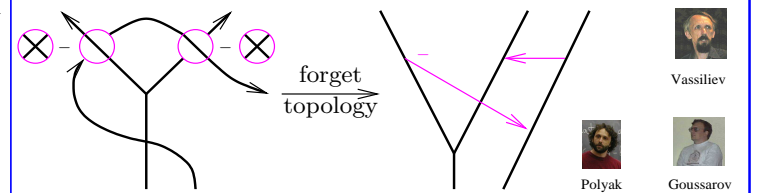


# Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots, Page 2

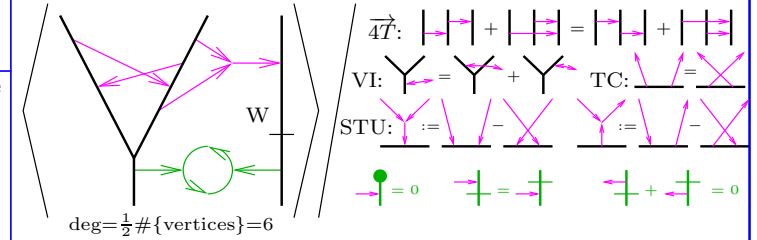
**Knot-Theoretic statement.** There exists a homomorphic expansion  $Z$  for trivalent w-tangles. In particular,  $Z$  should respect  $R4$  and intertwine annulus and disk unzips:



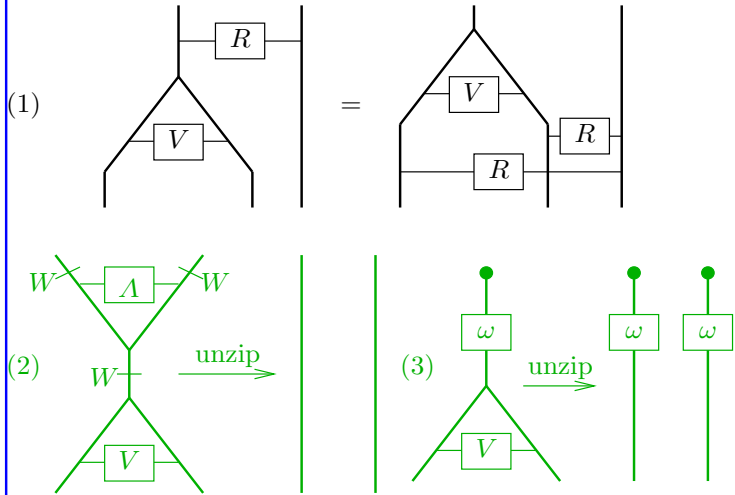
From wTT to  $\mathcal{A}^w$ :  $gr_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}$ :



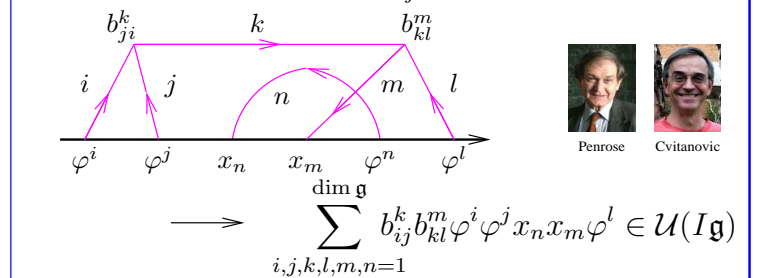
w-Jacobi diagrams and  $\mathcal{A}$ .  $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow \uparrow \uparrow)$  is



**Diagrammatic statement.** Let  $R = \exp \uparrow \uparrow \in \mathcal{A}^w(\uparrow \uparrow)$ . There exist  $\omega \in \mathcal{A}^w(\uparrow)$  and  $V \in \mathcal{A}^w(\uparrow \uparrow)$  so that



**Diagrammatic to Algebraic.** With  $(x_i)$  and  $(\varphi^j)$  dual bases of  $\mathfrak{g}$  and  $\mathfrak{g}^*$  and with  $[x_i, x_j] = \sum b_{ij}^k x_k$ , we have  $\mathcal{A}^w \rightarrow \mathcal{U}$  via



**Algebraic statement.** With  $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$ , with  $c : \hat{\mathcal{U}}(I\mathfrak{g}) \rightarrow \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$  the obvious projection, with  $S$  the antipode of  $\hat{\mathcal{U}}(I\mathfrak{g})$ , with  $W$  the automorphism of  $\hat{\mathcal{U}}(I\mathfrak{g})$  induced by flipping the sign of  $\mathfrak{g}^*$ , with  $r \in \mathfrak{g}^* \otimes \mathfrak{g}$  the identity element and with  $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$  there exist  $\omega \in \hat{S}(\mathfrak{g}^*)$  and  $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$  so that

(1)  $V(\Delta \otimes 1)(R) = R^{13} R^{23} V$  in  $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$   
 (2)  $V \cdot SWV = 1$       (3)  $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

**Unitary  $\iff$  Algebraic.** The key is to interpret  $\hat{\mathcal{U}}(I\mathfrak{g})$  as tangential differential operators on  $\text{Fun}(\mathfrak{g})$ :

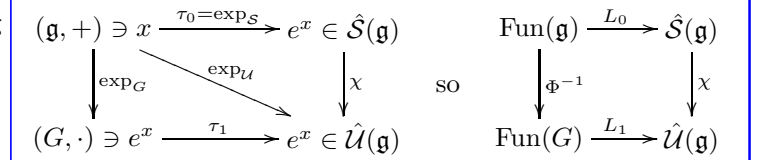
- $\varphi \in \mathfrak{g}^*$  becomes a multiplication operator.
- $x \in \mathfrak{g}$  becomes a tangential derivation, in the direction of the action of  $\text{ad } x$ :  $(x\varphi)(y) := \varphi([x, y])$ .
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \rightarrow \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$  is "the constant term".

**Unitary  $\implies$  Group-Algebra.**  $\iint \omega_{x+y}^2 e^{x+y} \phi(x)\psi(y)$   
 $= \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x)\psi(y) \rangle = \langle V\omega_{x+y}, V e^{x+y} \phi(x)\psi(y)\omega_{x+y} \rangle$   
 $= \langle \omega_x \omega_y, e^x e^y V \phi(x)\psi(y)\omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x)\psi(y)\omega_x \omega_y \rangle$   
 $= \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x)\psi(y).$

**Unitary statement.** There exists  $\omega \in \text{Fun}(\mathfrak{g})^G$  and an (infinite order) tangential differential operator  $V$  defined on  $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$  so that

(1)  $V \widehat{e^{x+y}} = \widehat{e^x e^y} V$  (allowing  $\hat{\mathcal{U}}(\mathfrak{g})$ -valued functions)  
 (2)  $VV^* = I$       (3)  $V\omega_{x+y} = \omega_x \omega_y$

**Convolutions and Group Algebras** (ignoring all Jacobians). If  $G$  is finite,  $A$  is an algebra,  $\tau : G \rightarrow A$  is multiplicative then  $(\text{Fun}(G), \star) \cong (A, \cdot)$  via  $L : f \mapsto \sum f(a)\tau(a)$ . For Lie  $(G, \mathfrak{g})$ ,



**Group-Algebra statement.** There exists  $\omega^2 \in \text{Fun}(\mathfrak{g})^G$  so that for every  $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$  (with small support), the following holds in  $\hat{\mathcal{U}}(\mathfrak{g})$ : (shhh,  $\omega^2 = j^{1/2}$ )

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_{x+y}^2 e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_x^2 \omega_y^2 e^x e^y. \quad (\text{shhh, this is Duflo})$$

with  $L_0 \psi = \int \psi(x)e^x dx \in \hat{S}(\mathfrak{g})$  and  $L_1 \Phi^{-1} \psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$ . Given  $\psi_i \in \text{Fun}(\mathfrak{g})$  compare  $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$  and  $\Phi^{-1}(\psi_1 \star \psi_2)$  in  $\hat{\mathcal{U}}(\mathfrak{g})$ : (shhh,  $L_{0/1}$  are "Laplace transforms")

**Convolutions statement** (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let  $G$  be a finite dimensional Lie group and let  $\mathfrak{g}$  be its Lie algebra, let  $j : \mathfrak{g} \rightarrow \mathbb{R}$  be the Jacobian of the exponential map  $\exp : \mathfrak{g} \rightarrow G$ , and let  $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$  be given by  $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$ . Then if  $f, g \in \text{Fun}(G)$  are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

- We skipped...**
- The Alexander polynomial and Milnor numbers.
  - v-Knots, quantum groups and Etingof-Kazhdan.
  - u-Knots, Alekseev-Torossian, and BF theory and the successful and Drinfel'd associators.
  - The simplest problem hyperbolic geometry solves.