# Projectivization, w−Knots, Kashiwara−Vergne and Alekseev−Torossian

"An Algebraic Structure"

## The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

## The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

### Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

$$\tau(F):=RF^{21}e(-t)\text{ is an involution},\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$


Alekseev

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$

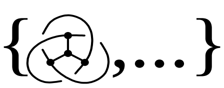

Torossian

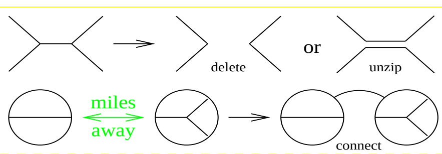
This is just a part of the Alekseev−Torossian work!

### So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

## Knotted Trivalent Graphs

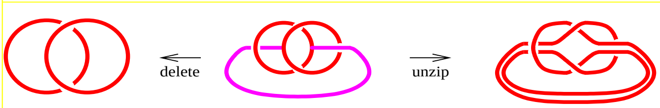
$\mathcal{O}(\triangle)=$ 




**Theorem.** KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.

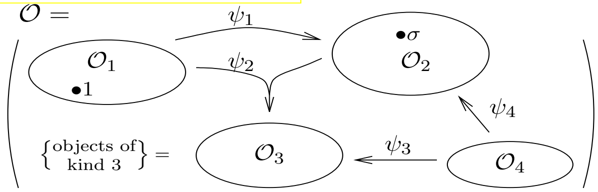
**Theorem.** $Z(\triangle)$ is equivalent to an associator $\Phi$.

### Algebraic Knot Theory




**Theorem.** $\{\text{ribbon knots}\}\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\!-\!\!\circ),\ d\gamma=\bigcirc\bigcirc\}$.

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.




- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

## Defining proj $\mathcal{O}$.

The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{array}{l}\text{formal differences of ob-}\\\text{jects "of the same kind"}\end{array}\right\}.$$

$$\text{Then }I^n:=\left\{\begin{array}{l}\text{all outputs of algebraic}\\\text{expressions at least }n\text{ of}\\\text{whose inputs are in }I\end{array}\right\},\text{ and}$$

$$\mathrm{proj}\,\mathcal{O}:=\bigoplus_{n\geq0}I^n/I^{n+1}\qquad\left(\begin{array}{l}\text{has same kinds and opera-}\\\text{tions, but different objects}\\\text{and axioms}\end{array}\right).$$

## Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type $n$ invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev


Goussarov

## Warmup Examples.

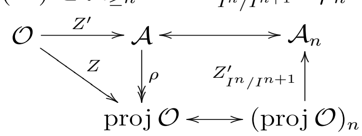
- The projectivization of a group is a graded associative algebra.
- A quandle: a set $Q$ with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

proj $Q$ is a graded Lie algebra: set $\bar{v}:=(v-1)$ (these generate $I$!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z}=(\bar{x}\wedge\bar{z})\wedge\bar{y}+\bar{x}\wedge(\bar{y}\wedge\bar{z}).$$

**An Expansion** is $Z\colon\mathcal{O}\to\mathrm{proj}\,\mathcal{O}$ s.t. $Z(I^n)\subset(\mathrm{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\mathrm{proj}\,\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:




Can you make this diagram less confusing?

**Homomorphic Expansions** are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.


X−S. Lin

**The Key Point.** If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/