

Dror Bar-Natan: Talks: Hanoi-0708: **Following Lin: Expansions for Groups**



Riverside, April 2000 Kyoto, September 2001
See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184-202.

- Vaughan's Hierarchy** (generalized, unauthorized)
- ☺ Computation
 - ☺ Formula
 - ☺ Proof
 - ☺ Theory
 - ☺ Dream

The Magnus and Exponential Expansions

$$Z_{1,2} : G_n = \left(\begin{array}{c} \text{free group} \\ \text{on} \\ X_1, \dots, X_n \end{array} \right) \rightarrow \hat{A}_n = \left(\begin{array}{c} \text{completed free} \\ \text{associative} \\ \text{algebra on} \\ x_1, \dots, x_n \end{array} \right)$$

by $X_i \mapsto 1 + x_i$ or e^{x_i}
 $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$ or e^{-x_i} .

What's "An Expansion"? A filtration-preserving isomorphism $Z : C(G) \rightarrow \mathcal{A}(G)$ where

$$I := \{ \sum a_i g_i : \sum a_i = 0 \} \subset CG$$

$$CG = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \dots$$

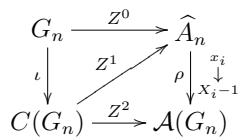
$$C(G) := \varprojlim_k CG/I^k \rightarrow \dots \rightarrow CG/I^2 \rightarrow CG/I \rightarrow 0$$

is filtered by $F_m C(G) := \varprojlim_{k>m} I^m/I^k$ and $\mathcal{A}(G) := \text{gr } C(G) = \bigoplus I^m/I^{m+1}$.
 So all expansions are "equivalent"

Think duals! $C(G)^*$ are "finite type invariants".
 $\mathcal{A}(G)^*$ are "weight systems".
 Z is a "universal finite type invariant".

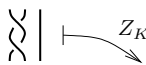
$Z_{1,2}$ are Expansions. With $Z^0 = Z_1$ or $Z^0 = Z_2$:

1. ι is automatic.
2. ρ is well-defined.
3. $Z^0|_{I^m} \subset F_m \mathcal{A}_n$.
4. Z^0 descends to Z^1 .
5. Define Z^2 .
6. ρ is surjective.
7. $\text{gr } Z^2$ is the identity.

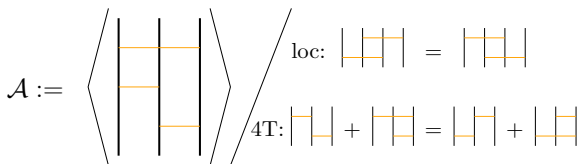
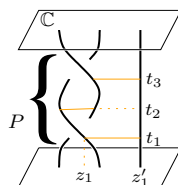


8. Z^2 is an isomorphism.
 9. ρ is an isomorphism.
- Everything generalizes, step 2 sometimes becomes tricky.

The Kontsevich Integral for Braids



$$\sum_{\substack{m, t_1 < \dots < t_m \\ P = \{(z_i, z'_i)\}}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz'_i}{z_i - z'_i}$$



M. Kontsevich

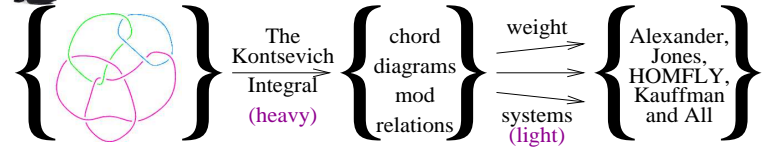
Which other groups / groupoids / categories have expansions?



Dror's Dream / Obsession:

"Unify" quantum groups – find one object that contains all.

Example: One invariant to rule them all:



Easy! Universal! A Morphism! Unique! An Isomorphism!

What is a "Quantum Group"? For now, a "deformation of the trivial" solution in $\mathcal{U}(\mathfrak{g})^{\otimes*}[[\hbar]]$ of the major equations:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta \otimes 1)R = R^{23}R^{13} \quad (1 \otimes \Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

Dror's Guess: A unified object exists; we'll need:

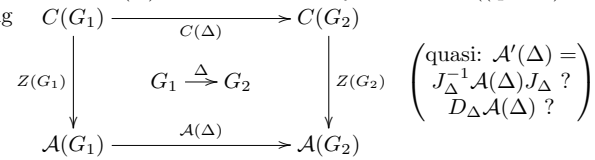
1. Expansions as in Lin / universal finite type invariants.
2. Naturality / functoriality.
3. Knotted graphs, especially trivalent.
4. Associators following Drinfel'd.
5. The work of Etingof and Kazhdan on bialgebras.
6. Virtual braids / knots / knotted graphs.
7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams.
 (and when construction ends, we'll dump the scaffolding)

Why care?
 Quantum groups computable invariants make!

Visit!
 katlas.org
Edit!

(Quasi?) Natural Expansions

$G \mapsto C(G)$ and $G \mapsto \mathcal{A}(G)$ are functors. Can you choose a ((quasi?) natural) Z satisfying



Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:

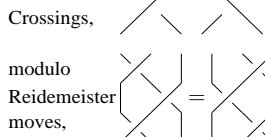


Note the relation:

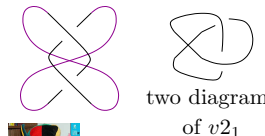
Virtual Braids

crossings are real, strands go virtual

Definition.

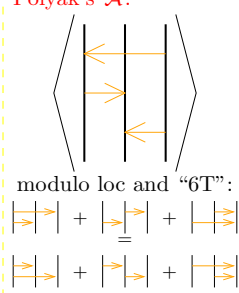


but the linkages between crossings are "virtual":



L. Kauffman

Polyak's A.



modulo loc and "6T":



M. Polyak



T. Ohtsuki

Lie bialgebras.

The \mathfrak{g} in a sum $\mathfrak{g} \oplus \mathfrak{g}^*$ which in itself is a Lie algebra with subalgebras \mathfrak{g} and \mathfrak{g}^* , and in which the tautological metric is invariant.

Why bother?

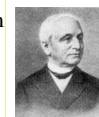
Their deformations are quantum groups, and their diagrammatic universalization is $\vec{\mathcal{A}}$.



Drinfel'd Etingof Kazhdan Haviv

Question Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

Dror's Guess: No, but the effort will be worthwhile.



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)