

Math 1352 Algebraic Knot Theory - The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}^2 \times \mathbb{C}$ (Fixed endpoints)

$$Z(B) = \oint \frac{Dp}{(2\pi i)^m} \prod_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'} \text{ in } \mathcal{A}(1_n) := \langle t^{ij} : |k| \neq j \leq n \rangle / \begin{matrix} t^{ij} = -t^{ji} \\ [t^{ij}, t^{kl}] = 0 \\ [t^{ij}, t^{ik} + t^{jk}] = 0 \end{matrix}$$

t_1, \dots, t_m
 $p = (z_1, z_1')$

Formal Connection to Curvature.

horizontal chords.
= "Chord diagrams for braids".

Let $\Omega \in \mathcal{A}(1_n)$ with $\deg \Omega = 1$.
 $\gamma: [0, 1] \times I \rightarrow M$ induces
 $\phi: \Delta^m = \{0 \leq t_1 \leq \dots \leq t_m \leq 1\} \rightarrow M^m$.
 Set $\text{hol}_\gamma(\Omega) = P \circ \exp \int \Omega = \oint_{\Delta^m} \phi^* \Omega^m$

where $\Omega^m := \pi_1^* \Omega \wedge \dots \wedge \pi_m^* \Omega$

Theorem 2. If $F_\Omega := d\Omega + \Omega \wedge \Omega = 0$, then $\text{hol}_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ .

Proof 2. Let $\Gamma: I_s \times I_t \rightarrow M$, $\Phi: I_s \times \Delta^m \rightarrow M^m$.
 By Stokes',
 $\int_{\Delta^m} \Phi^* \Omega^m - \int_{\Delta^m_0} \Phi^* \Omega^m = \int_{I \times \Delta^m} d\Phi^* \Omega^m - \int_{I \times \Delta^m} \Phi^* \Omega^m =: A_m - B_m$

Now
 $A_m = \sum_{k=1}^m (-1)^{k+1} \int_{I \times \Delta^m} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* d\Omega \wedge \dots \wedge \pi_m^* \Omega$

and
 $B_m = \int_{I \times [t_1=0]} \Phi^* \Omega^m \pm \int_{I \times [t_m=1]} \Phi^* \Omega^m + \sum_{k=1}^{m-1} (-1)^k \int_{I \times [t_k=t_{k+1}]} \Phi^* \Omega^m$
 $= \sum_{k=1}^{m-1} (-1)^k \int_{I \times \Delta^{m-1}} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* (\Omega \wedge \Omega) \wedge \dots \wedge \pi_{m-1}^* \Omega$

and now $\sum A_m = \sum B_m$ by telescopic summation & $F_\Omega = 0$.

The KZ connection.

$M = \mathbb{C}^n \setminus \{\text{diagonals}\}$, $A = \mathcal{A}(1_n)$,

and $\Omega = \sum_{i < j} t^{ij} W_{ij}$ where $W_{ij} = \frac{dz_i - dz_j}{z_i - z_j} \stackrel{\text{locally}}{=} d \log(z_i - z_j)$

Compute $F_\Omega = d\Omega + \Omega \wedge \Omega$: $dW_{ij} = 0$ so $d\Omega = 0$.

$\Omega \wedge \Omega = \sum_{\substack{i < j \\ k < l}} t^{ij} t^{kl} W_{ij} \wedge W_{kl} = A + B + C$ where

$A = C = 0$ as $[t^{ij}, t^{kl}] = 0$ if $\{|i, j, k, l|\} = 2$ or 4 and

$B = \sum_{\alpha < \beta < \gamma} [t^{\alpha\beta}, t^{\beta\gamma}] W_{\alpha\beta} \wedge W_{\beta\gamma} + \text{cyclic perms}$

$= \sum_{\alpha < \beta < \gamma} \gamma^{\alpha\beta\gamma} (W_{\alpha\beta} \wedge W_{\beta\gamma} + \text{cyc perms}) = 0$ by Arnold's identity

Note: by 4T,
 $[t^{\alpha\beta}, t^{\beta\gamma}] = 0$
 $\gamma^{\alpha\beta\gamma} = \text{diagram}$

Simply take in theorem 2,
 $\gamma =$ the braid
 and

$\Omega =$ the KZ connection.

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