Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

Dror Bar−Natan, Strasbourg 2011

This is an overview with too many and not enough details. I apologize.

http://www.math.toronto.edu/~drorbn/Talks/Strasbourg-1109/

Foots & refs on PDF version

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

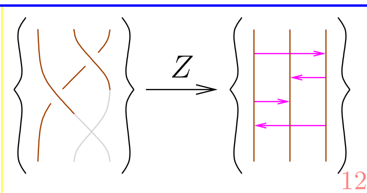
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle\ker\mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.

Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filterred $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



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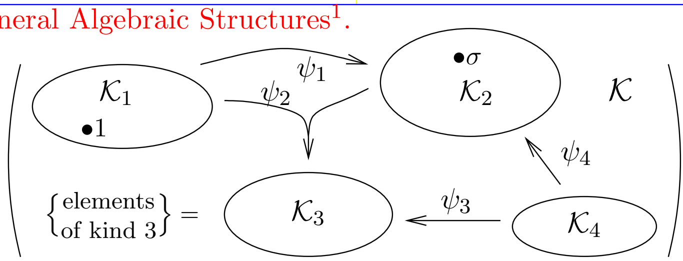
General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works! 14

Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Leibniz[2] algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

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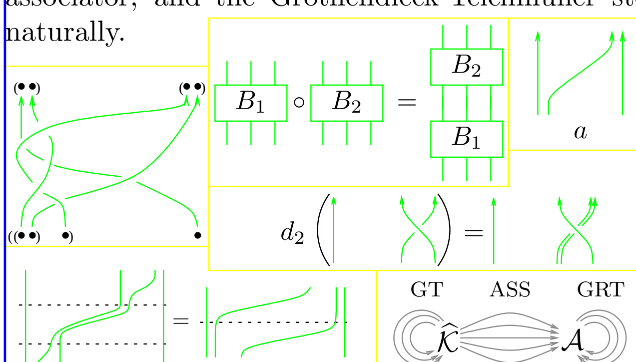
Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[3] arises naturally.



Drinfel'd [Dr1,2]

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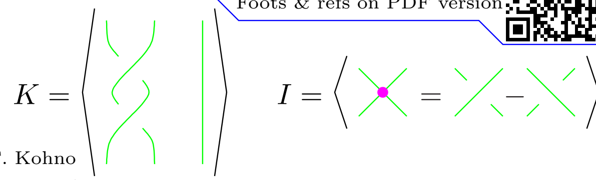
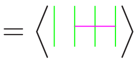
Example 1.

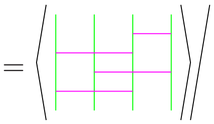
T. Kohno



$(K/I^{m+1})^\star = \text{(invariants of type } m) =: \mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1}$ $\quad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle = $ 

$\ker\mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle\text{4T relations}\rangle$

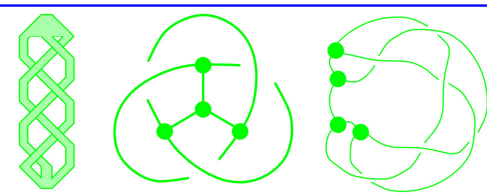
$\mathcal{A} = A_n = \left(\text{horizontal chord diagrams mod 4T}\right) = $ 

Z: universal finite type invariant, the Kontsevich integral. 9

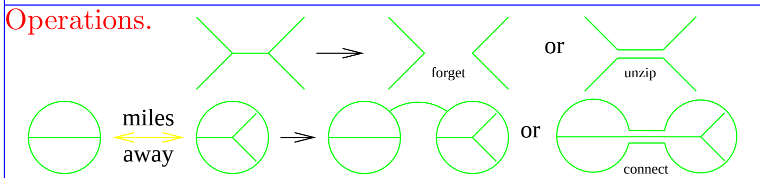
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

Example 5 - Knotted Trivalent Graphs.

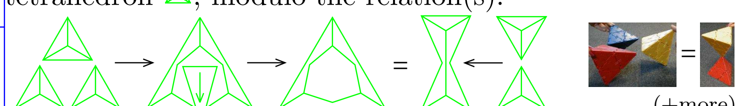
D. Thurston [Th]



Operations.

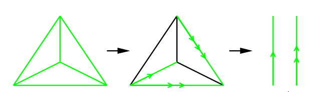


Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):



(+more)

Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

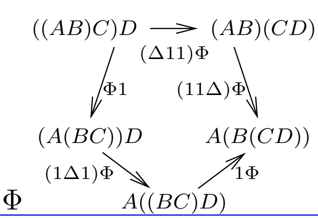


$:= \Phi \in \mathcal{A}(\uparrow_3)$

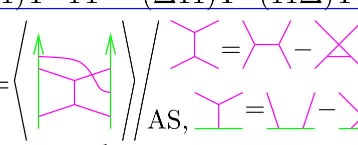
A $\mathcal{U}(\mathfrak{g})$-Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

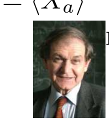


$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$

$\mathcal{A}(\uparrow_2) :=$ 

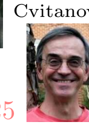
$(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$

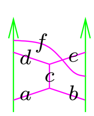
$\xrightarrow[\text{metrized } \mathfrak{g},\ \mathfrak{g} = \langle X_a\rangle]{\text{Given a}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$



Penrose

Cvitanovic





$$\to \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

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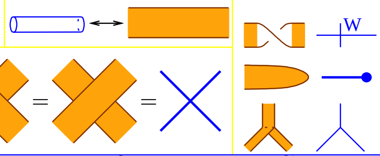
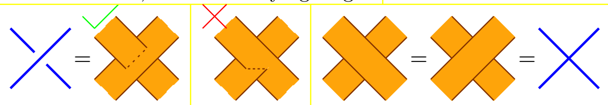
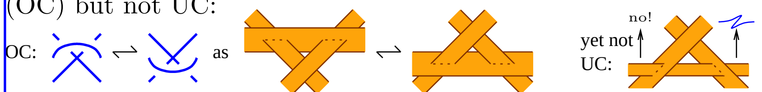
Facts and Dreams About v-Knots and Etingof-Kazhdan, 2

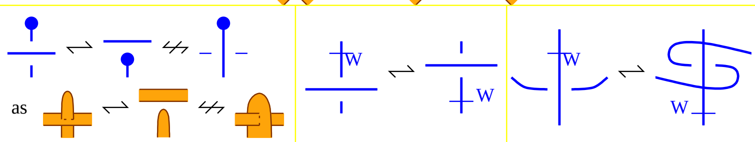
Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".

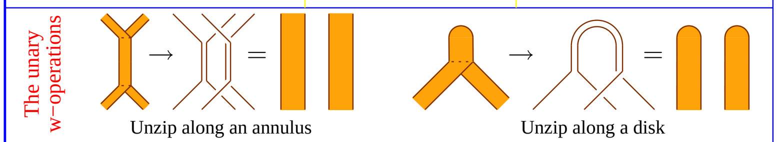




The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



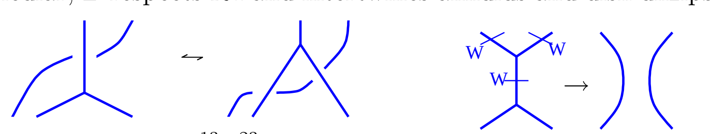




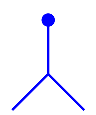
Trivalent w-Tangles. 30

$$wTT = PA\left\langle \text{w-generators} \,\middle|\, \text{w-relations} \,\middle|\, \text{unary w-operations} \right\rangle = CA\left\langle \text{same w/o} \times \right\rangle$$

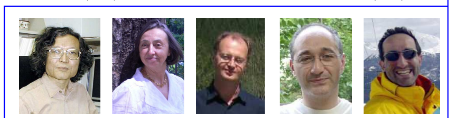
Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:



$V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow_3)$ $\quad VV^* = I$ in $\mathcal{A}^w(\uparrow_2)$



$V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow_2)$



Kashiwara-Vergne-Alekseev-Enriquez-Torrosian

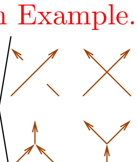
Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F\in \mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that 33

$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y).$$

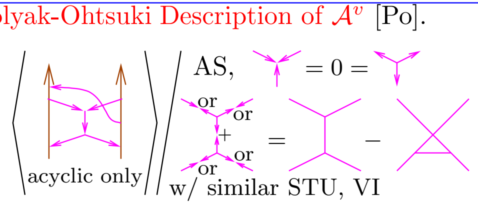
Theorem. That's equivalent to a homomorphic expansion for wTT.

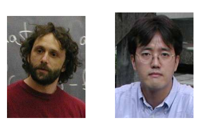
The Main Example.

$$vTT = PA\left\langle \text{[generators]} \,\middle|\, \text{R234, VR234, D, yet not UC, OC} \,\middle|\, \text{unzips}\right\rangle = \widetilde{CA}\left\langle \text{same w/o} \times \right\rangle$$



The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po].





Polyak Ohtsuki

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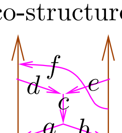
$\mathcal{A}^v$ pairs with Lie bialgebras. Let $\mathfrak{g}_+$ be a Lie bialgebra with basis X_a, bracket $[\cdot,\cdot]$, cobracket δ, dual $\mathfrak{g}_- = \mathfrak{g}_+^\star$, dual basis X^a for $\mathfrak{g}_-$, double $\mathfrak{g} = \mathfrak{g}_+\oplus\mathfrak{g}_-$, structure constants $[X_a,X_b] = \sum b^c_{ab}X_c$ and co-structure constants $\delta(X_a) = \sum c^{bc}_a X_b\otimes X_c$. Then



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} b^c_{de}c^{ba}_c X_a X^d X_f\otimes X_b X^f X^e \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$$

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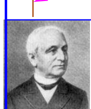


"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Forbidden Theorem [EK, Ha, ?]. There exists a homomorphic expansion Z for vTT.

Why Forbidden (to me)?

- Minor statement details may be off.
- No fully written proof.
- I don't understand the proof.
- There isn't yet a knot-theoretic view of the proof, like there is in the w-case. 42

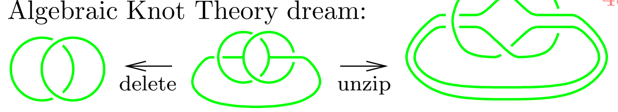
Etingof

Haviv

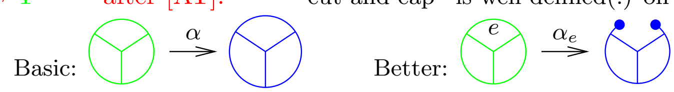
Kazhdan

Why Should We Care?

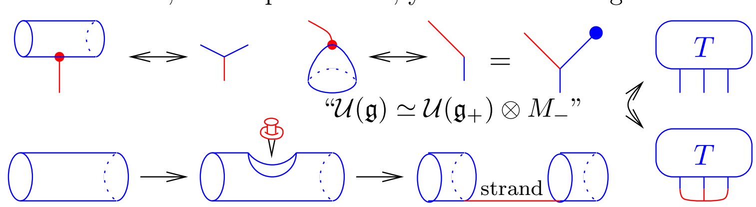
- A gateway into the forbidden territory of "quantum groups".
- Abstractly more pleasing: We study the things, and not just their representations.
- $\mathcal{A}^v$ is sometimes easier than $\mathcal{A}^u$: Alexander, say, arises easily from the 2D Lie algebra[4].
- Potentially, $\mathcal{A}^v$ has many more "internal quotients" than there are Lie bialgebras. What are they and what are the corresponding theories?
- My old[5] Algebraic Knot Theory dream: 45

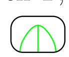
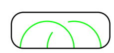


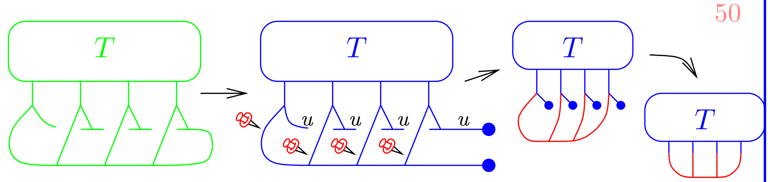
$V\to\Phi^{\text{1-loop}}$ after [AT]. "cut and cap" is well-defined(!) on $\mathcal{K}^u$

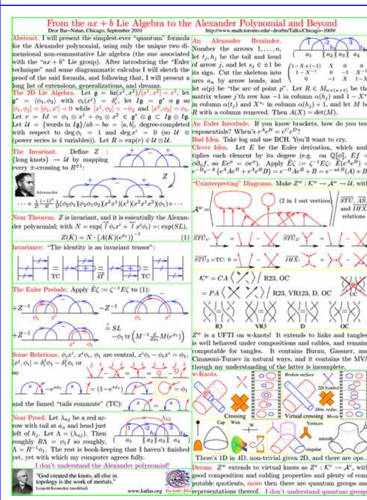


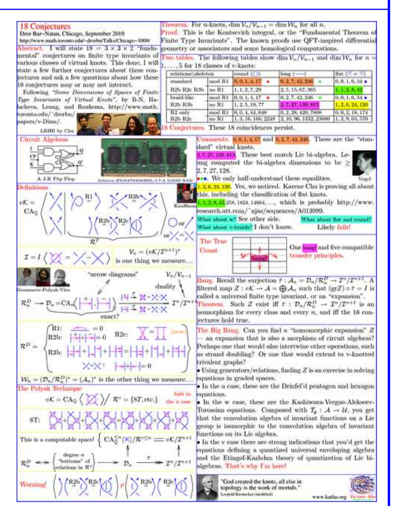
$\Phi\to V$ after [AET]. In $\mathcal{K}^{\overline{w}}$ allow tubes and strands and tube-strand vertices, allow "punctures", yet allow no "tangles".



The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V). With T any classical tangle, esp.  or , consider the "sled" 50







Alexander is easy! In Chicago, [BN4] Many kinds of virtuals!

Help Needed!

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. A Leibniz algbera is a Lie algebra minus the anti-symmetry of the bracket; I have previously erroneously asserted that here $\mathcal{A}(K)$ is Lie; however see the comment by Conant attached to this talk's video page.
3. See my paper [BN1] and my talk/handout/video [BN3].
4. See [BN5] and my talk/handout/video [BN4].
5. Not so old and not quite written up. Yet see [BN2].

References

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[BN2] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN3] D. Bar-Natan, *Braids and the Grothendieck-Teichmüller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[BN4] D. Bar-Natan, *From the $ax + b$ Lie Algebra to the Alexander Polynomial and Beyond,* talk given in Chicago on September 11, 2010, `http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/`.

[BN5] D. Bar-Natan, *Finite Type Invariants of w-Knotted Objects: From Alexander to Kashiwara and Vergne,* in preparation, online at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

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Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention PvB, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minutes) Example: parenthesized braids and horizontal associators.
6. (6 minutes) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (8 minutes) Example: wKO's and the Kashiwara-Vergne equations.
8. (12 minutes) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minutes) wKO's, uKO's, and Alekseev-Enriquez-Torossian.