Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

Dror Bar−Natan at Swiss Knots 2011

This is an overview with too many and not enough details. I apologize.

http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/
Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.


Peter Lee

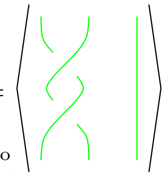
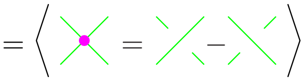
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filterred $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

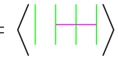
Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 1.

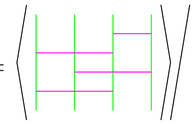

T. Kohno

$K =$  $I =$ 

$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$

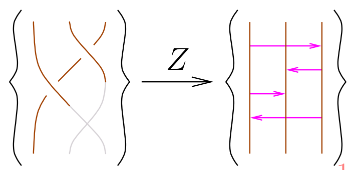
$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle =$ 

$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle 4\text{T relations}\rangle$

$\mathcal{A} = A_n = \left(\text{horizontal chord diagrams mod 4T}\right) =$  /4T

Z: universal finite type invariant, the Kontsevich integral. 9

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



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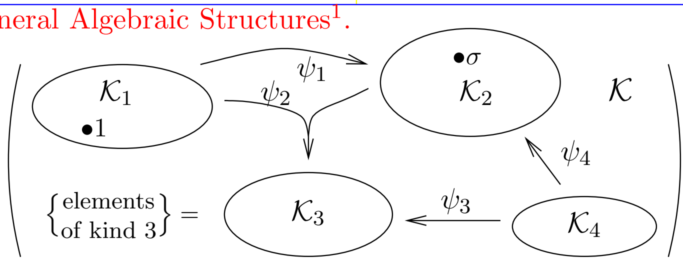
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

General Algebraic Structures[1].

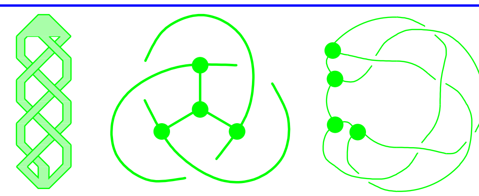


- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations. 14

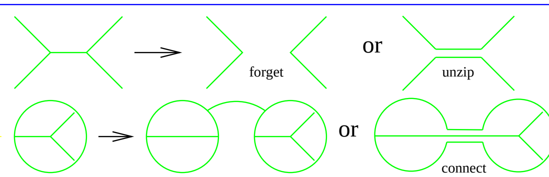
All still works!

Example 5 - Knotted Trivalent Graphs.


D. Thurston [Th]



Operations.



Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Leibniz[2] algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

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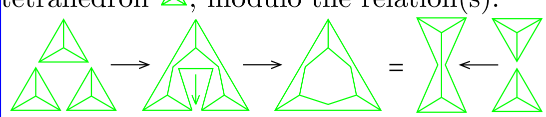
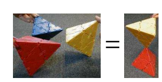
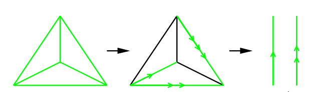
Presentation. KTG is generated by ribbon twists and the tetrahedron $\triangle$, modulo the relation(s):



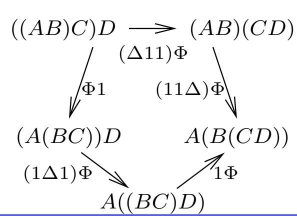

(+more)

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.


$:= \Phi \in \mathcal{A}(\uparrow_3)$

A $\mathcal{U}(\mathfrak{g})$-Associator:

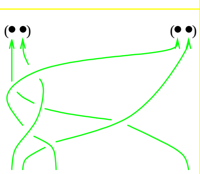
$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

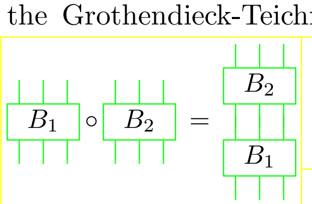


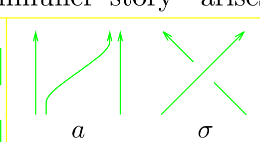
satisfying the "pentagon",

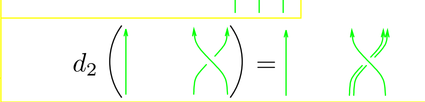
$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$

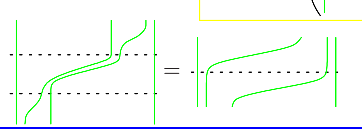
Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[3] arises naturally.

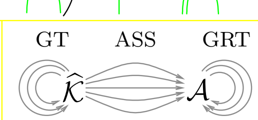













Drinfel'd [Dr1,2]

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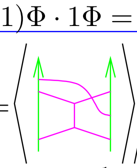
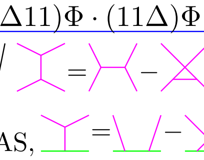
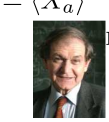
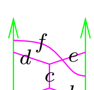
$\mathcal{A}(\uparrow_2) :=$  / AS,  $\xrightarrow[\text{metrized } \mathfrak{g},\ \mathfrak{g} = \langle X_a\rangle]{\text{Given a}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$

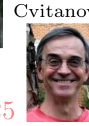
$(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$


Penrose



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

Cvitanovic



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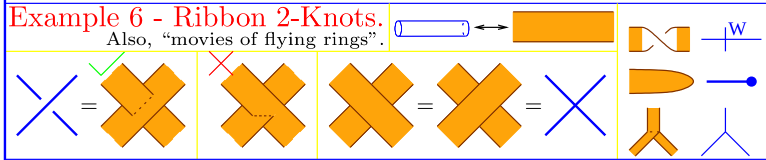
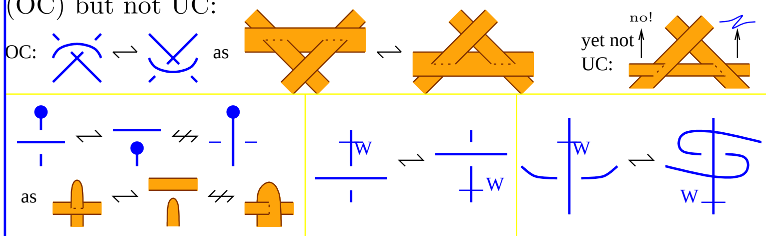
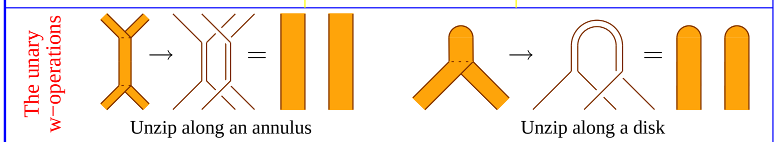
Facts and Dreams About v−Knots and Etingof−Kazhdan, 2

Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

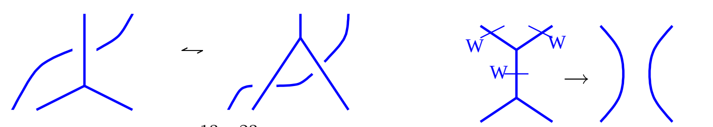




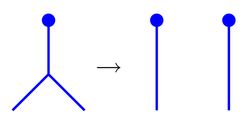
Trivalent w-Tangles. 30

$$wTT = PA\left\langle \text{w-generators} \,\middle|\, \text{w-relations} \,\middle|\, \text{unary w-operations} \right\rangle = CA\left\langle \text{same w/o} \times \right\rangle$$

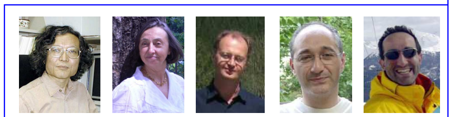
Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:



$V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow_3)$ $\quad VV^* = I$ in $\mathcal{A}^w(\uparrow_2)$



$V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow_2)$



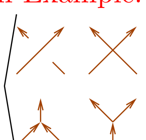
Kashiwara-Vergne-Alekseev-Enriquez-Torrosian

Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F\in \mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

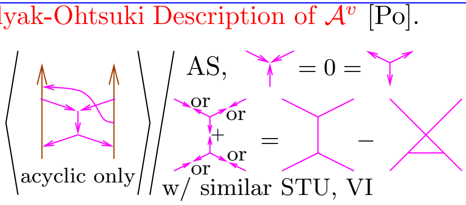
$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y). \qquad 33$$

Theorem. That's equivalent to a homomorphic expansion for wTT.

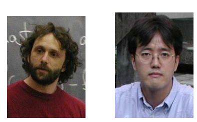
The Main Example.

vTT= PA⟨  | R234, VR234, D, yet not UC, OC | unzips ⟩ $=\widetilde{CA}\langle$ same w/o × ⟩

The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po].



$\mathcal{A}^v \simeq$ ⟨ acyclic only | AS, ..., w/ similar STU, VI ⟩



Polyak Ohtsuki

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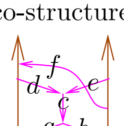
$\mathcal{A}^v$ pairs with Lie bialgebras. Let $\mathfrak{g}_+$ be a Lie bialgebra with basis X_a, bracket $[\cdot,\cdot]$, cobracket δ, dual $\mathfrak{g}_- = \mathfrak{g}_+^\star$, dual basis X^a for $\mathfrak{g}_-$, double $\mathfrak{g} = \mathfrak{g}_+\oplus\mathfrak{g}_-$, structure constants $[X_a, X_b] = \sum b_{ab}^c X_c$ and co-structure constants $\delta(X_a) = \sum c_a^{bc} X_b\otimes X_c$. Then



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} b_{de}^c c_c^{ba} X_a X^d X_f \otimes X_b X^f X^e \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$$

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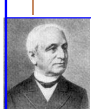


"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Forbidden Theorem [EK, Ha, ?]. There exists a homomorphic expansion Z for vTT.

Etingof

Haviv

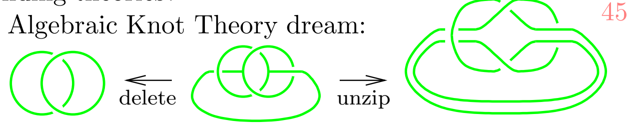
Kazhdan

Why Forbidden (to me)?

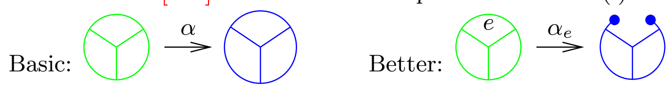
- Minor statement details may be off.
- No fully written proof.
- I don't understand the proof.
- There isn't yet a knot-theoretic view of the proof, like there is in the w-case. 42

Why Should We Care?

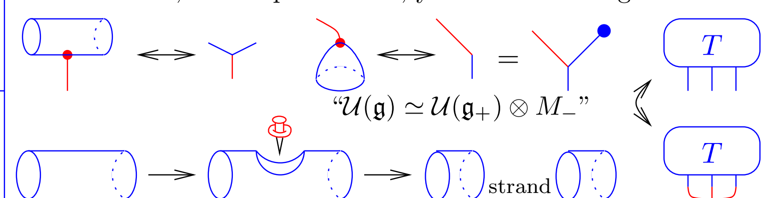
- A gateway into the forbidden territory of "quantum groups".
- Abstractly more pleasing: We study the things, and not just their representations.
- $\mathcal{A}^v$ is sometimes easier than $\mathcal{A}^u$: Alexander, say, arises easily from the 2D Lie algebra[4].
- Potentially, $\mathcal{A}^v$ has many more "internal quotients" than there are Lie bialgebras. What are they and what are the corresponding theories? 45
- My old[5] Algebraic Knot Theory dream:



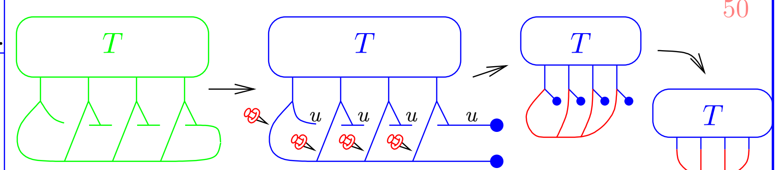
$V \to \Phi^{1\text{-loop}}$ after [AT]. "cut and cap" is well-defined(!) on $\mathcal{K}^u$

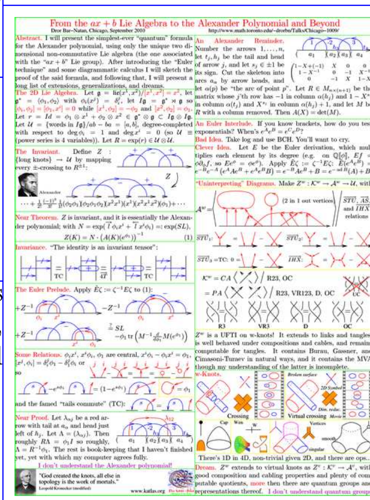


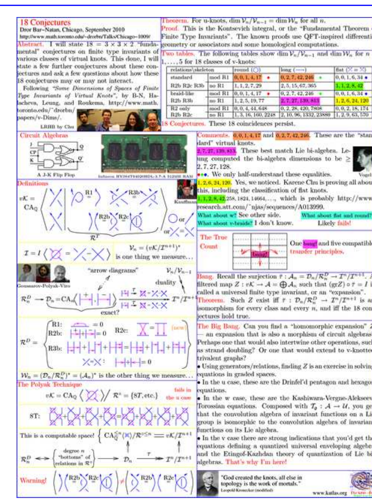
$\Phi \to V$ after [AET]. In $\mathcal{K}^{\overline{w}}$ allow tubes and strands and tube-strand vertices, allow "punctures", yet allow no "tangles".



The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V). With T any classical tangle, esp. [diagram] or [diagram], consider the "sled" 50







Alexander is easy! In Chicago, [BN4] Many kinds of virtuals!

Help Needed!

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. A Leibniz algbera is a Lie algebra minus the anti-symmetry of the bracket; I have previously erroneously asserted that here $\mathcal{A}(K)$ is Lie; however see the comment by Conant attached to this talk's video page.
3. See my paper [BN1] and my talk/handout/video [BN3].
4. See [BN5] and my talk/handout/video [BN4].
5. Not so old and not quite written up. Yet see [BN2].

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, Braid groups and explicit solutions of the Kashi-waraVergne equations,* Pub. Math. de L'IHES **112-1** (2010) 143–189, arXiv:arXiv:0903.4067.

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the YangBaxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmüller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN3] D. Bar-Natan, *Braids and the Grothendieck-Teichmüller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[BN4] D. Bar-Natan, *From the $ax + b$ Lie Algebra to the Alexander Polynomial and Beyond,* talk given in Chicago on September 11, 2010, `http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/`.

[BN5] D. Bar-Natan, *Finite Type Invariants of w-Knotted Objects: From Alexander to Kashiwara and Vergne,* in preparation, online at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

[Dr1,2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457 and *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with Gal*$(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

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[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.

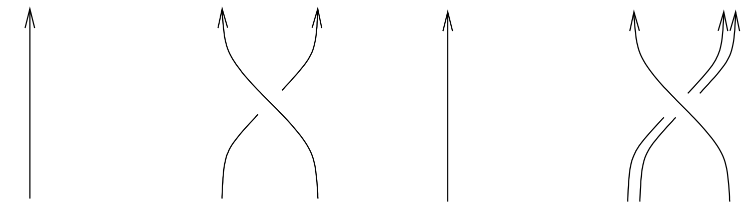
[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.

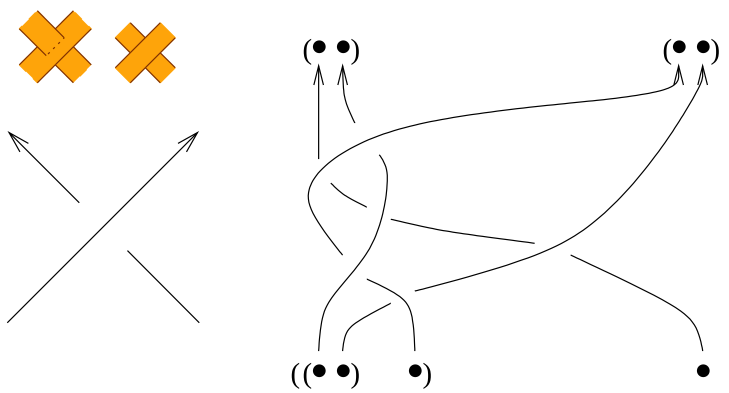
[Th] D. P. Thurston, *The Algebra of Knotted Trivalent Graphs and Turaev's Shadow World,* Geometry & Topology Monographs **4** (2002) 337-362, arXiv:math.GT/0311458.

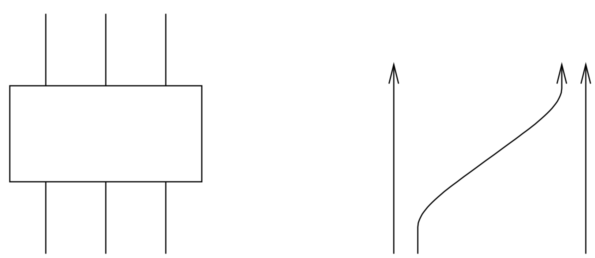
Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention PvB, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minutes) Example: parenthesized braids and horizontal associators.
6. (6 minutes) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (8 minutes) Example: wKO's and the Kashiwara-Vergne equations.
8. (12 minutes) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minutes) wKO's, uKO's, and Alekseev-Enriquez-Torossian.

Scratch Work

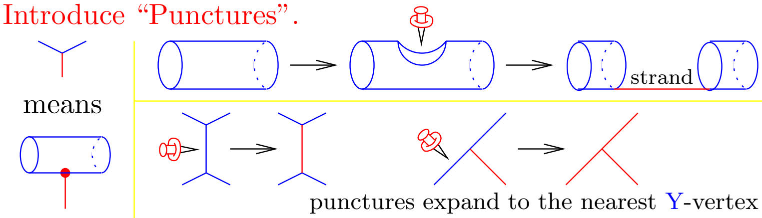


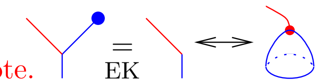




$\Phi \to V$ after [AET]. The generators of $\mathcal{K}^w$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V).

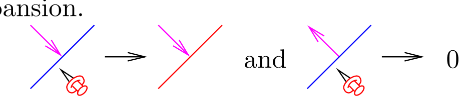
Introduce "Punctures".

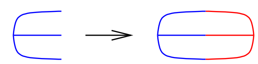


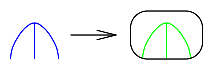
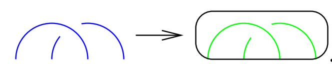
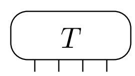
Note. 

$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices, yet allow only "compact" knots — no runs to ∞.

$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z: 

$\Longleftarrow$  (makes sense, and CA ops can be emulated)

 and , so enough to write any 

. Here go: