

Following Lin: Expansions for Groups



Riverside, April 2000



Kyoto, September 2001

See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184-202.

The Magnus and Exponential Expansions

$$Z_{1,2} : G_n = \left(\begin{array}{c} \text{free group} \\ \text{on} \\ X_1, \dots, X_n \end{array} \right) \rightarrow \hat{A}_n = \left(\begin{array}{c} \text{completed free} \\ \text{associative} \\ \text{algebra on} \\ x_1, \dots, x_n \end{array} \right)$$

by $X_i \mapsto 1 + x_i$ or e^{x_i}

$$X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots \text{ or } e^{-x_i}.$$

What's "An Expansion"? A filtration-preserving isomorphism $Z : C(G) \rightarrow \mathcal{A}(G)$ where

$$I := \{ \sum a_i g_i : \sum a_i = 0 \} \subset \mathbb{C}G$$

$$\mathbb{C}G = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \dots$$

$$C(G) := \varprojlim_k \mathbb{C}G/I^k \rightarrow \dots \rightarrow \mathbb{C}G/I^2 \rightarrow \mathbb{C}G/I \rightarrow \mathbb{C}$$

is filtered by $F_m C(G) := \varprojlim_{k>m} I^m/I^k$ and

$$\mathcal{A}(G) := \text{gr } C(G) = \hat{\bigoplus} I^m/I^{m+1}.$$

Think duals! $C(G)^*$ are "finite type invariants".

$\mathcal{A}(G)^*$ are "weight systems".

Z is a "universal finite type invariant".

$Z_{1,2}$ are Expansions. With $Z^0 = Z_1$ or $Z^0 = Z_2$:

1. ι is automatic.

2. ρ is well-defined.

3. $Z^0|_{I^m} \subset F_m \mathcal{A}_n$.

4. Z^0 descends to Z^1 .

5. Define Z^2 .

6. ρ is surjective.

7. $\text{gr } Z^2$ is the identity.

$$\begin{array}{ccc} G_n & \xrightarrow{Z^0} & \hat{A}_n \\ \iota \downarrow & \nearrow Z^1 & \downarrow \rho \\ C(G_n) & \xrightarrow{Z^2} & \mathcal{A}(G_n) \end{array}$$

$\begin{array}{c} x_i \\ \downarrow \\ x_i^{-1} \end{array}$

8. Z^2 is an isomorphism.

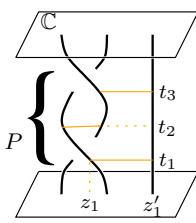
9. ρ is an isomorphism.

Everything generalizes, step 2 sometimes becomes tricky.

The Kontsevich Integral for Braids

$$\text{crossing} \xrightarrow{Z_K} \text{crossing}$$

$$\sum_{\substack{m, t_1 < \dots < t_m \\ P = \{(z_i, z'_i)\}}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz'_i}{z_i - z'_i}$$



$$\mathcal{A} := \left(\text{braids} \right) / \left(\begin{array}{l} \text{loc: } \text{||||} = \text{||||} \\ \text{4T: } \text{||||} + \text{||||} = \text{||||} + \text{||||} \end{array} \right)$$

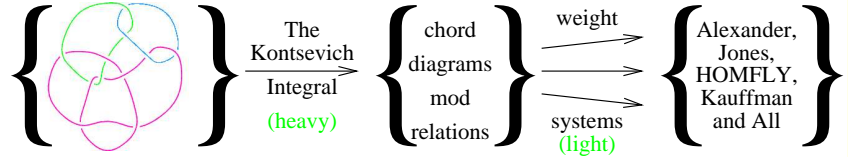


M. Kontsevich

Dror's Dream / Obsession:

"Unify" quantum groups – find one object that contains them all.

Example: One invariant to rule them all:



Easy! Universal! A Morphism! Unique! An Isomorphism!

What is a "Quantum Group"? For now, a "deformation of the trivial" solution in $\mathcal{U}(\mathfrak{g})^{\otimes*}[[\hbar]]$ of the major equations:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta \otimes 1)R = R^{23}R^{13} \quad (1 \otimes \Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

Dror's Guess: A unified object exists; we'll need:

1. Expansions as in Lin / universal finite type invariants.

2. Naturality / functoriality.

3. Knotted graphs, especially trivalent.

4. Associators following Drinfel'd.

5. The work of Etingof and Kazhdan on bialgebras.

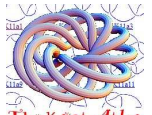
6. Virtual braids / knots / knotted graphs.

7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams.

(and when construction ends, we'll dump the scaffolding)

Why care?

Quantum groups computable invariants make!



The Knot Atlas - Anyone Can Edit

Visit!

katlas.org

Edit!

(Quasi?) Natural Expansions

$G \mapsto C(G)$ and $G \mapsto \mathcal{A}(G)$ are functors. Can you choose a ((quasi?) natural) Z satisfying

$$\begin{array}{ccc} C(G_1) & \xrightarrow{C(\Delta)} & C(G_2) \\ \downarrow z(G_1) & \searrow G_1 \xrightarrow{\Delta} G_2 & \downarrow z(G_2) \\ \mathcal{A}(G_1) & \xrightarrow{\mathcal{A}(\Delta)} & \mathcal{A}(G_2) \end{array}$$

Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:

$$\text{crossing} \rightarrow \text{crossing} \mid \text{ or } \text{crossing} \mid = \text{||} \text{ or } \text{||}$$

Virtual Braids

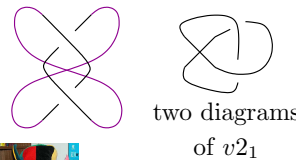
crossings are real, strands go virtual

Definition.

Crossings,

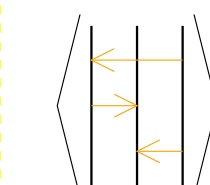


modulo Reidemeister moves, but the linkages between crossings are "virtual":



L. Kauffman

Polyak's $\vec{\mathcal{A}}$.



modulo loc and "6T":



M. Polyak



T. Ohtsuki

Lie bialgebras.

The \mathfrak{g} in a sum $\mathfrak{g} \oplus \mathfrak{g}^*$ which in itself is a Lie algebra with subalgebras \mathfrak{g} and \mathfrak{g}^* , and in which the tautological metric is invariant.

Why bother?

Their deformations are quantum groups, and their diagrammatic universalization is $\vec{\mathcal{A}}$.



Drinfel'd Etingof Kazhdan Haviv

Question Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

Dror's Guess: No, but the effort will be worthwhile.



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)