Braids and the Grothendieck−Teichmuller Group

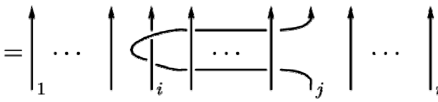
Dror Bar−Natan, Toronto, January 2011, http://www.math.toronto.edu/~drorbn/Talks/Toronto−100110/

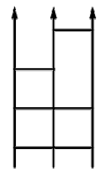
Abstract. The "Grothendieck-Teichmuller Group" (**GT**) appears as a "depth certificate" in many recent works — "we do A to B, apply the result to C, and get something related to **GT**, therefore it must be interesting". Interesting or not, in my talk I will explain how **GT** arose first, in Drinfel'd's work on associators, and how it can be used to show that "every bounded-degree associator extends", that "rational associators exist", and that "the pentagon implies the hexagon"*.

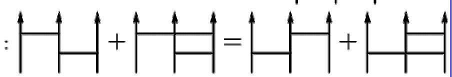
In a nutshell: the filtered tower of braid groups (with bells and whistles attached) is isomorphic to its associated graded, but the isomorphism is neither canonical nor unique — such an isomorphism is precisely the thing called "an associator". But the set of isomorphisms between two isomorphic objects **always** has two groups acting simply transitively on it — the group of automorphisms of the first object acting on the right, and the group of automorphisms of the second object acting on the left. In the case of associators, that first group is what Drinfel'd calls the Grothendieck-Teichmuller group **GT**, and the second group, isomorphic but not canonically to the first and denoted **GRT**, is the one several recent works seem to refer to.

Almost everything I will talk about is in my old paper "On Associators and the Grothendieck-Teichmuller Group I", also at arXiv:q-alg/9606021.

Baby(?) Example. PB_n: pure braids; $I \subset \mathbb{Q}PB_n$ the augmentation ideal; $B^{(m)} = \mathbb{Q}PB_n/I^{m+1}$ (filtered!); $\hat{B} = \varprojlim B^{(m)}$ (filtered!). Then $\operatorname{gr} B^{(m)} = C^{(m)}$ and then $\operatorname{gr}\hat{B} = \hat{C}$ where $C = \langle t^{ij} = t^{ji} \colon [t^{ij}, t^{kl}] = [t^{ij}, t^{ik}+t^{jk}] = 0\rangle$, so $B^{(m)}$ and $\hat{B}$ are isomorphic to $C^{(m)}$ and $\hat{C}$, but not canonically. Me not know that the groups **GT** and **GRT** here have been analyzed.

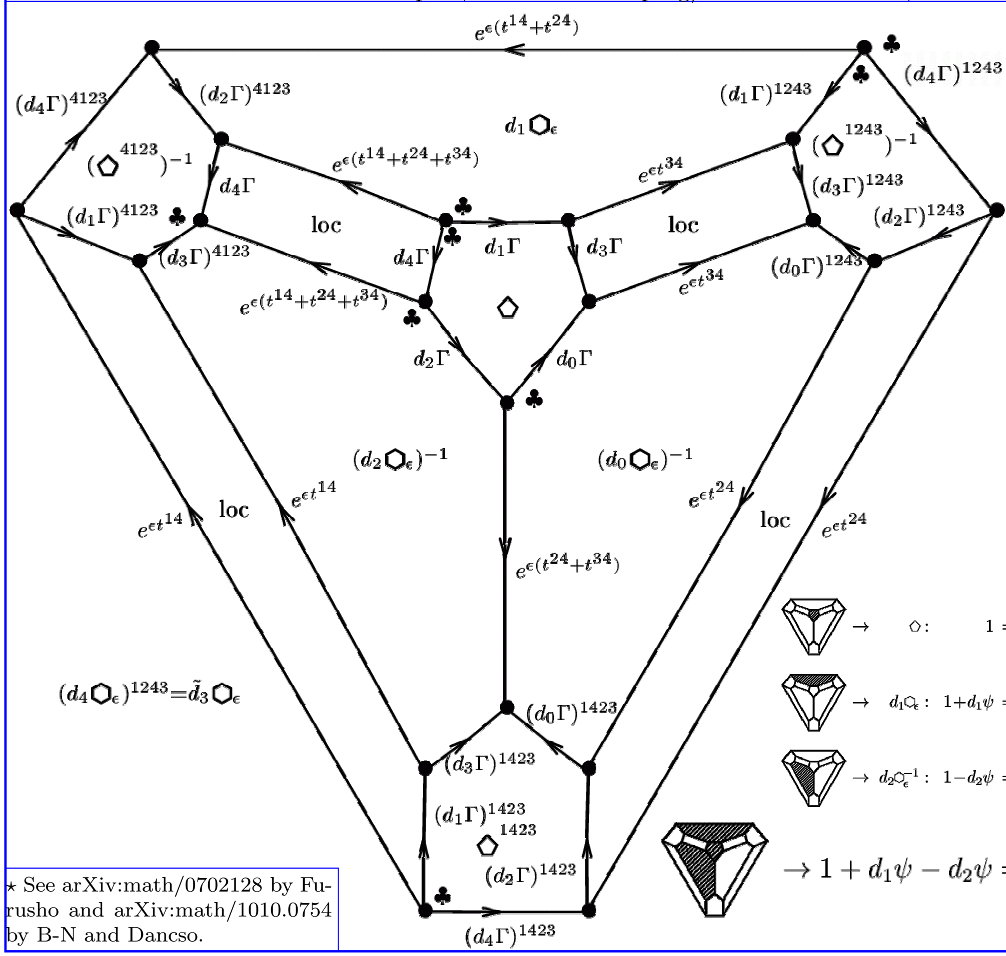
$\sigma^{ij} =$ 

$t^{13}t^{13}t^{12}t^{23} \longleftrightarrow$ 

$4T:$ 

Main Theorem. The projection $\mathbf{ASSO}^{(m)} \to \mathbf{ASSO}^{(m-1)}$ is surjective. (yey!)

Sketch. Given $\mathbf{ASSO}^{(m)} \neq \emptyset$ (hard, analytic), sufficient is surjectivity of $\mathbf{GRT}^{(m)} \to \mathbf{GRT}^{(m-1)}$, enough is surjectivity of $\mathfrak{grt}^{(m)} \to \mathfrak{grt}^{(m-1)}$, polyhedron on left use, little homological algebra too.



$\to$ ⬠: $1 = d_4\Gamma d_2\Gamma d_0\Gamma (d_3\Gamma)^{-1}(d_1\Gamma)^{-1}$,

$\to$ $d_1⬠_\epsilon$: $1 + d_1\psi = d_1\Gamma e^{\epsilon t^{34}}((d_1\Gamma)^{1243})^{-1} e^{\epsilon(t^{14}+t^{24})}(d_2\Gamma)^{4123} e^{-\epsilon(t^{14}+t^{24}+t^{34})}$

$\to$ $d_2⬠_\epsilon^{-1}$: $1 - d_2\psi =$ (product around shaded area).

$\to 1 + d_1\psi - d_2\psi =$ (product around shaded area)

$\star$ See arXiv:math/0702128 by Furusho and arXiv:math/1010.0754 by B-N and Dancso.

by successive approximations presents no problems. For this we introduce the following modification $\mathrm{GRT}(k)$ of the group $\mathrm{GT}(k)$. We denote by $\mathrm{GRT}_1(k)$ the set of all $g \in \mathrm{Fr}_k(A, B)$ such that

$$g(B, A) = g(A, B)^{-1}, \quad (5.12)$$

$$g(C, A)g(B, C)g(A, B) = 1 \quad \text{for } A + B + C = 0, \quad (5.13)$$

$$A + g(A, B)^{-1}Bg(A, B) + g(A, C)^{-1}Cg(A, C) = 0 \quad \text{for } A + B + C = 0, \quad (5.14)$$

$$g(X^{12}, X^{23} + X^{24})g(X^{13} + X^{23}, X^{34}) = g(X^{23}, X^{34})g(X^{12} + X^{13}, X^{24} + X^{34})g(X^{12}, X^{23}), \quad (5.15)$$

where the X^{ij} satisfy (5.1). $\mathrm{GRT}_1(k)$ is a group with the operation

$$(g_1 \circ g_2)(A, B) = g_1(g_2(A, B)Ag_2(A, B)^{-1}, B) \cdot g_2(A, B). \quad (5.16)$$

On $\mathrm{GRT}_1(k)$ there is an action of k^*, given by $\tilde{g}(A, B) = g(c^{-1}A, c^{-1}B)$, $c \in k^*$. The semidirect product of k^* and $\mathrm{GRT}_1(k)$ we denote by $\mathrm{GRT}(k)$. The Lie algebra $\mathfrak{grt}_1(k)$ of the group $\mathrm{GRT}_1(k)$ consists of the series $\psi \in \mathfrak{fr}_k(A, B)$ such that

$$\psi(B, A) = -\psi(A, B), \quad (5.17)$$

$$\psi(C, A) + \psi(B, C) + \psi(A, B) = 0 \quad \text{for } A + B + C = 0, \quad (5.18)$$

$$[B, \psi(A, B)] + [C, \psi(A, C)] = 0 \quad \text{for } A + B + C = 0, \quad (5.19)$$

$$\psi(X^{12}, X^{23} + X^{24}) + \psi(X^{13} + X^{23}, X^{34}) = \psi(X^{23}, X^{34}) + \psi(X^{12} + X^{13}, X^{24} + X^{34}) + \psi(X^{12}, X^{23}), \quad (5.20)$$

where the X^{ij} satisfy (5.1). A commutator $\langle\ ,\ \rangle$ in $\mathfrak{grt}_1(k)$ is of the form

$$\langle\psi_1, \psi_2\rangle = [\psi_1, \psi_2] + D_{\psi_2}(\psi_1) - D_{\psi_1}(\psi_2), \quad (5.21)$$

where $[\psi_1, \psi_2]$ is the commutator in $\mathfrak{fr}_k(A, B)$ and D_ψ is the derivation of $\mathfrak{fr}_k(A, B)$ given by $D_\psi(A) = [\psi, A]$, $D_\psi(B) = 0$. The algebra $\mathfrak{grt}_1(k)$ is

Proposition 5.1. *The action of* $\mathrm{GT}(k)$ *on* $M(k)$ *is free and transitive.*

Proof. If $(\mu, \varphi) \in M(k)$ and $(\overline{\mu}, \overline{\varphi}) \in M(k)$, then there is exactly one f such that $\overline{\varphi}(A, B) = f(\varphi(A, B)e^A\varphi(A, B)^{-1}, e^B) \cdot \varphi(A, B)$. We need to show that $(\lambda, f) \in \mathrm{GT}(k)$, where $\lambda = \overline{\mu}/\mu$. We prove (4.10). Let G_n be the semidirect product of S_n and $\exp \mathfrak{a}_n^k$. Consider the homomorphism $B_n \to G_n$ that takes σ_i into

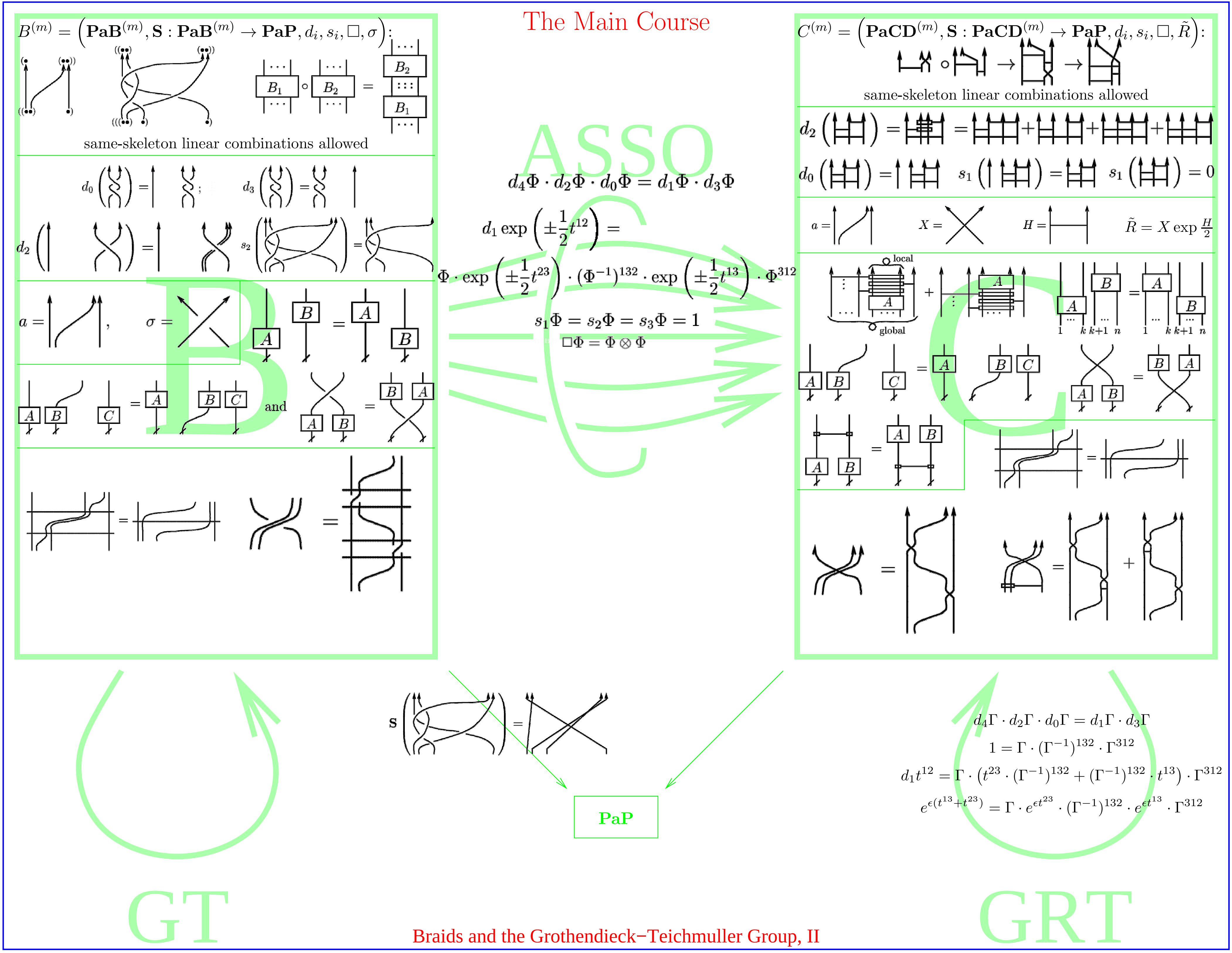
$$\varphi(X^{1i} + \cdots + X^{i-1,i}, X^{i,i+1})^{-1}\sigma^{i,i+1}e^{\mu X^{i,i+1}/2}\varphi(X^{1i} + \cdots + X^{i-1,i}, X^{i,i+1}),$$

where $\sigma^{ij} \in S_n$ transposes i and j. It induces a homomorphism $K_n \to \exp \mathfrak{a}_n^k$, and therefore a homomorphism $\alpha_n \colon K_n(k) \to \exp \mathfrak{a}_n^k$, where $K_n(k)$ is the k-pro-unipotent completion of K_n. It is easily shown that the left- and right-hand sides of (4.10) have the same images in $\exp \mathfrak{a}_4^k$. It remains to prove that α_n is an isomorphism. The algebra Lie $K_n(k)$ is topologically generated by the elements ξ_{ij}, $1 \le i < j \le n$, with defining relations obtained from (4.7)–(4.9) by substituting $x_{ij} = \exp \xi_{ij}$. The principal parts of these relations are the same as in (5.1), while $(\alpha_n)_*(\xi_{ij}) = \mu X^{ij} +$ {lower terms}, where $(\alpha_n)_* \colon \mathrm{Lie}\, K_n(k) \to \mathfrak{a}_n^k$ is induced by the homomorphism α_n. Therefore α_n is an isomorphism, i.e., (4.10) is proved. (4.3) is obvious. To prove (4.4), we can interpret it in terms of K_3 and argue as in the proof of (4.10), or, what is equivalent, make the substitution

$$X_1 = e^A, \quad X_2 = e^{-A/2}\varphi(B, A)e^B\varphi(B, A)^{-1}e^{A/2}, \quad X_3 = \varphi(C, A)e^C\varphi(C, A)^{-1}, \quad (5.4)$$

where $A + B + C = 0$. ●

From Drinfel'd's *On quasitriangular Quasi-Hopf algebras and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J. **2** (1991) 829–860.


The Main Course
$B^{(m)} = \left(\mathbf{PaB}^{(m)}, \mathbf{S} : \mathbf{PaB}^{(m)} \to \mathbf{PaP}, d_i, s_i, \square, \sigma\right)$:
same-skeleton linear combinations allowed
$a =$, $\sigma =$
and
$C^{(m)} = \left(\mathbf{PaCD}^{(m)}, \mathbf{S} : \mathbf{PaCD}^{(m)} \to \mathbf{PaP}, d_i, s_i, \square, \tilde{R}\right)$:
same-skeleton linear combinations allowed
$a =$ $X =$ $H =$ $\tilde{R} = X \exp \frac{H}{2}$
local
global
ASSO
$d_4\Phi \cdot d_2\Phi \cdot d_0\Phi = d_1\Phi \cdot d_3\Phi$
$d_1 \exp\left(\pm\frac{1}{2}t^{12}\right) =$
$\Phi \cdot \exp\left(\pm\frac{1}{2}t^{23}\right) \cdot (\Phi^{-1})^{132} \cdot \exp\left(\pm\frac{1}{2}t^{13}\right) \cdot \Phi^{312}$
$s_1\Phi = s_2\Phi = s_3\Phi = 1$
$\square\Phi = \Phi \otimes \Phi$
B
C
PaP
$d_4\Gamma \cdot d_2\Gamma \cdot d_0\Gamma = d_1\Gamma \cdot d_3\Gamma$
$1 = \Gamma \cdot (\Gamma^{-1})^{132} \cdot \Gamma^{312}$
$d_1 t^{12} = \Gamma \cdot \left(t^{23} \cdot (\Gamma^{-1})^{132} + (\Gamma^{-1})^{132} \cdot t^{13}\right) \cdot \Gamma^{312}$
$e^{\epsilon(t^{13}+t^{23})} = \Gamma \cdot e^{\epsilon t^{23}} \cdot (\Gamma^{-1})^{132} \cdot e^{\epsilon t^{13}} \cdot \Gamma^{312}$
GT
GRT
Braids and the Grothendieck−Teichmuller Group, II