Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Toronto, March 2013

ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Toronto-1303

15 Minutes on Algebra

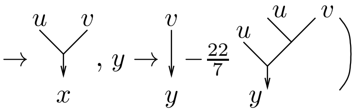
Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \left(\begin{matrix}\text{anti-symmetry}\\ \text{Jacobi}\end{matrix}\right)$$

... with the obvious bracket.

$M_{1/2}(u,v;x,y) = \Big\{\lambda = \Big(x \to$  $\Big)\dots\Big\}$

Operations $M_{1/2} \to M_{1/2}$. newspeak!

Tail Multiply tm^{uv}_w is $\lambda \mapsto \lambda /\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!/ tm^{uw}_u = tm^{vw}_v /\!/ tm^{uv}_u$.

Head Multiply hm^{xy}_z is $\lambda \mapsto (\lambda \setminus \{x,y\}) \cup (z \to \text{bch}(\lambda_x, \lambda_y))$, where

$$\text{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\text{bch}(\text{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \text{bch}(\alpha,\text{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!/ hm^{xz}_x = hm^{yz}_y /\!/ hm^{xy}_x$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda /\!/ RC_u^{\lambda_x}$, where $C_u^{-\gamma}\colon FL \to FL$ is the substitution $u \to e^{-\gamma} u e^{\gamma}$, or more precisely,

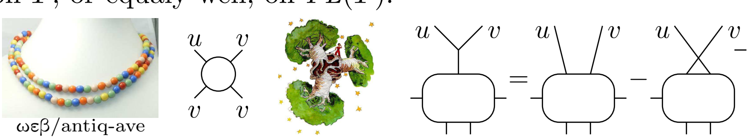
$$C_u^{-\gamma}\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and RC_u^γ is the inverse of that. Note that $C_u^{\text{bch}(\alpha,\beta)} = C_u^{\alpha /\!/ RC_u^{-\beta}} /\!/ C_u^\beta$ and hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z,$$

and $tm^{uv}_w /\!/ C_w^{\gamma /\!/ tm^{uv}_w} = C_u^{\gamma /\!/ RC_v^{-\gamma}} /\!/ C_v^\gamma /\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^x v^x$", $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$.

Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:

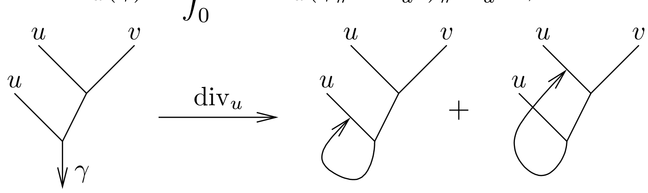


ωεβ/antiq-ave

Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!/ RC_u^\gamma,$$

where $J_u(\gamma) := \int_0^1 ds\, \text{div}_u(\gamma /\!/ RC_u^{s\gamma}) /\!/ C_u^{-s\gamma}$, and




Alekseev

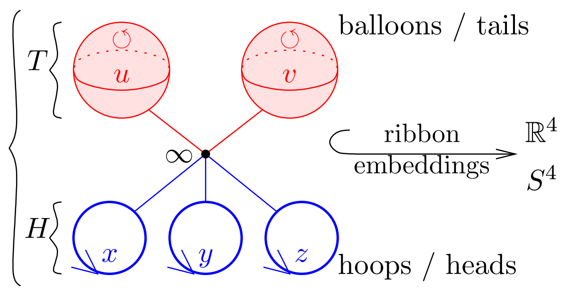

Torossian

Theorem Blue. All blue identities still hold.

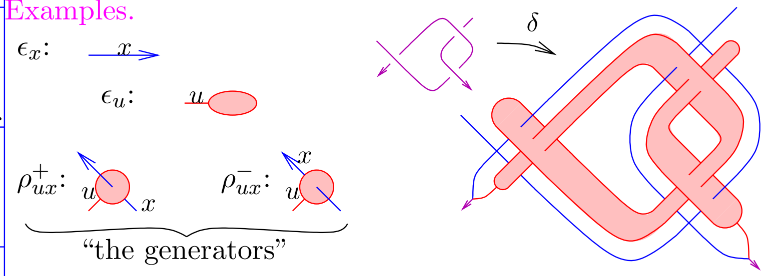
Merge Operation. $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

15 Minutes on Topology

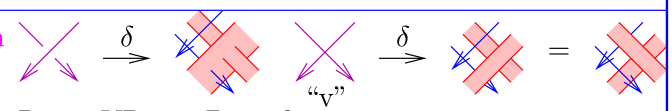
$\mathcal{K}^{bh}(T;H)$. "Ribbon-knotted balloons and hoops"



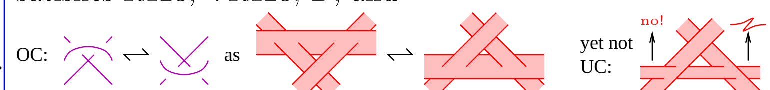
Examples.



More on δ

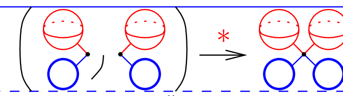


satisfies R123, VR123, D, and

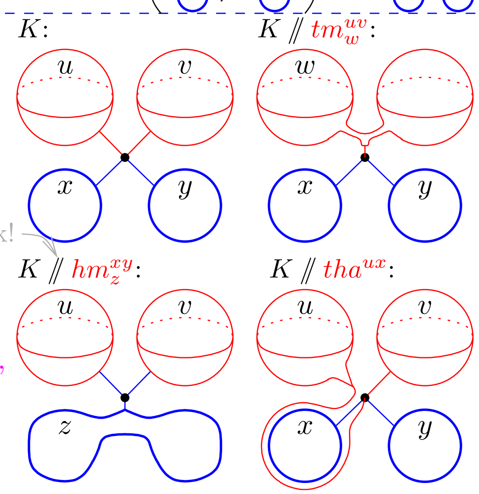


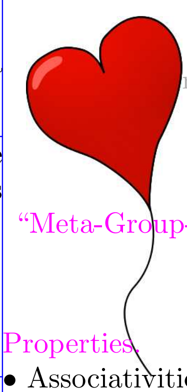
- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel is as above, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

Operations Punctures & Cuts | Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.



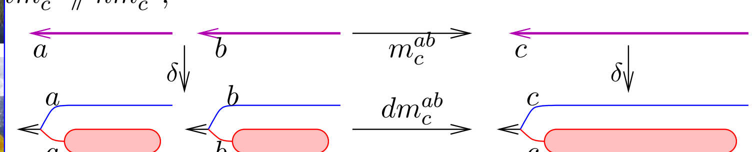


"Meta-Group-Action"

Properties.

- Associativities: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^x v^x$": $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z$.

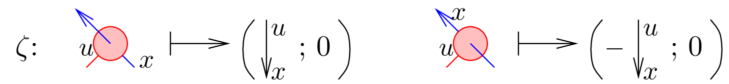
Tangle concatenations $\to \pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} /\!/ tm^{ab}_c /\!/ hm^{ab}_c$,



Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

Trees and Wheels and Balloons and Hoops: Why I Care

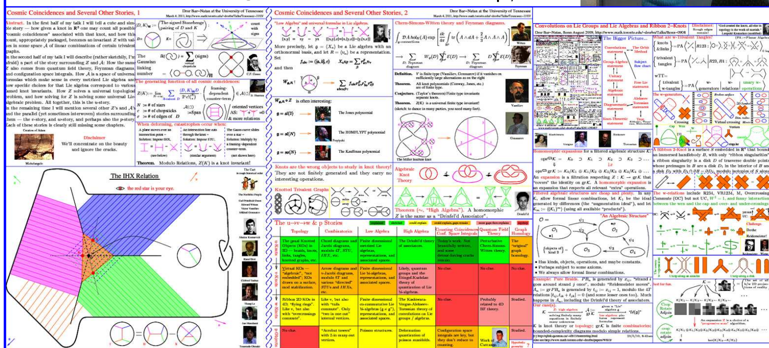
The Invariant ζ. Set $\zeta(\epsilon_x)=(x\to 0;0)$, $\zeta(\epsilon_u)=(();0)$, and



Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$.

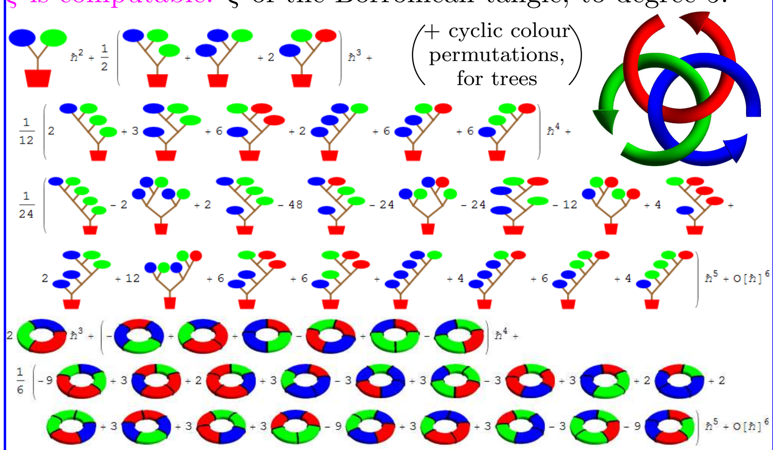
(... and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko



See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

ζ is computable! ζ of the Borromean tangle, to degree 5:



Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau: FL(T)\to \text{Fun}(\oplus_T\mathfrak{g}\to\mathfrak{g})$ and $\tau: CW(T)\to\text{Fun}(\oplus_T\mathfrak{g})$. Together, $\tau: M(T;H)\to \text{Fun}(\oplus_T\mathfrak{g}\to\oplus_H\mathfrak{g})$, and hence

$$e^\tau: M(T;H)\to\text{Fun}(\oplus_T\mathfrak{g}\to\mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\text{hol}_{\gamma_x}(A)\in\mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B)\in\mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u.


Cattaneo

Loose Conjecture. For $\gamma\in\mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}Be^{\int B\wedge F_A}\prod_u e^{\mathcal{O}_{\gamma_u}(B))}\bigotimes_x \text{hol}_{\gamma_x}(A)=e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

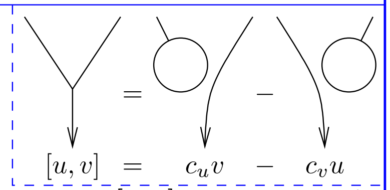
www.katlas.org The Knot Atlas Anyone Can Edit

May class: ωεβ/aarhus

Class next year: ωεβ/1350

Paper in progress: ωεβ/kbh

The β quotient is M divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R=\mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R\otimes T$ with central R and with $[u,v]=c_uv-c_vu$ for $u,v\in T$. Then $FL\to L_\beta$ and $CW\to R$. Under this,



$$\mu\to((\lambda_x);\omega)\quad\text{with } \lambda_x=\sum_{u\in T}\lambda_{ux}ux,\quad \lambda_{ux},\omega\in R,$$

$$\text{bch}(u,v)\to\frac{c_u+c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u+e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\gamma=\sum\gamma_v v$ then with $c_\gamma:=\sum\gamma_v c_v$,

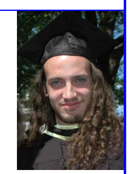
$$u /\!/ RC_u^\gamma=\left(1+c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u-c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$

$\text{div}_u\,\gamma=c_u\gamma_u$, and $J_u(\gamma)=\log\left(1+\frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right)$, so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x\to\lambda_{ux});\omega)$, set $c_x:=\sum_v c_v\lambda_{vx}$, replace $\lambda_{ux}\to\alpha_{ux}:=c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega\to e^\omega$, use $t_u=e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(T;H)$ be

$$\left\{\begin{array}{c|ccc|l} \omega & x & y & \cdots & \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot & \omega \text{ and the } \alpha_{ux}\text{'s are rational functions in} \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & \text{variables } t_u, \text{ one for each } u\in T. \\ \vdots & \cdot & \cdot & \cdot & \end{array}\right\},$$


With Selmani, ωεβ/meta

$$tm^{uv}_w: \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array},\qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z: \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

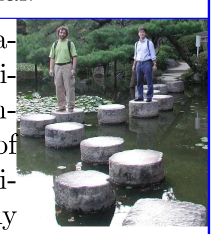
$$tha^{ux}: \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon:=1+\alpha$, $\langle\alpha\rangle:=\sum_v\alpha_v$, and $\langle\gamma\rangle:=\sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux}:=\begin{array}{c|c} 1 & x \\ \hline u & t_u-1\end{array}\qquad R^-_{ux}:=\begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1}-1\end{array}.$$

On long knots, ω is the Alexander polynomial!

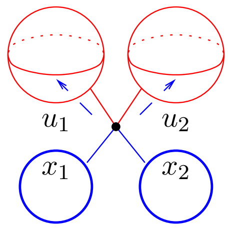
Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with an algebraic categorification, it is this one!*



See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

Trees and Wheels and Balloons and Hoops - Extras / Recycling

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c),(u_i),(x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1. $*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longtitude":



Not computable!
(but nearly)

$$(\Pi,(u,\ldots),(x,\ldots)) \mapsto (\Pi * \langle\bar{u}\rangle/(u=x\bar{u}x^{-1}),(\bar{u},\ldots),(x,\ldots))$$

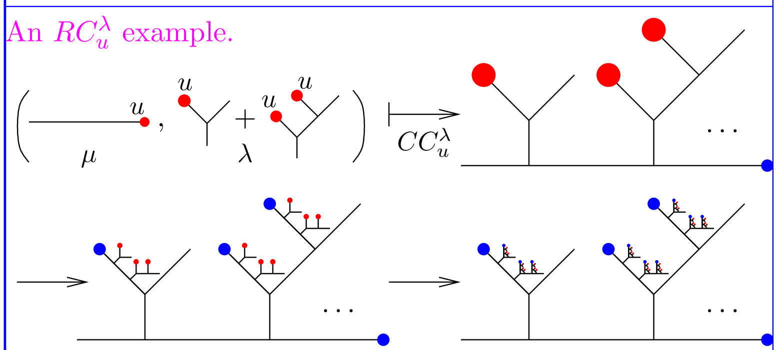
Failure #0. Can we write the x's as free words in the u's? If $x=uv$, compute $x /\!\!/ tha^{ux}$:

$$x=uv \to \bar{u}v = u^x v = u^{\bar{u}v}v = u^{u^x v}v = u^{u^{u^x v}v}v = \cdots$$

Why ODEs? **Q.** Find f s.t. $f(x+y)=f(x)f(y)$.
A. $\frac{df(s)}{ds} = \frac{d}{d\epsilon}f(s+\epsilon) = \frac{d}{d\epsilon}f(s)f(\epsilon) = f(s)C$.
Now solve this ODE using Picard's theorem or power series.

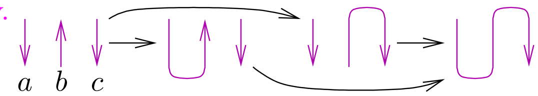
Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

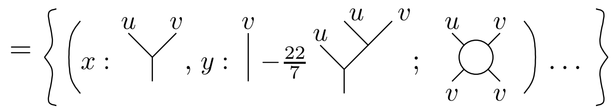
An RC_u^λ example.



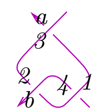
("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

Meta-associativity.
$m^{ab}_a /\!\!/ m^{ac}_a$
$= m^{bc}_b /\!\!/ m^{ab}_a$.





I mean business!

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T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[2, 1, 1] // dm[4, b, b] //
    dm[1, a, a] // dm[3, a, a];
ζ[{5}] /. {w_LW :> (Deg[w] + 1) ! w,
  w_CW :> Deg[w] ! w}

μ[CWS[-[a], -2 [ab], -3 [aab] - 3 [abb],
  -4 [aaab] + 42 [aabb] - 60 [abab] - 4 [abbb],
  -5 [aaaab] + 110 [aaabb] - 180 [aabab] +
   110 [aabbb] - 180 [ababb] - 5 [abbbb]],
 h[b] LS[2 ⟨a⟩, 0, -24 ⟨aab⟩,
    -60 ⟨aaab⟩ + 60 ⟨aabb⟩, -120 ⟨aaaab⟩ +
     900 ⟨aaabb⟩ + 360 ⟨aabab⟩ - 120 ⟨aabbb⟩] +
  h[a] LS[-2 ⟨a⟩ + 2 ⟨b⟩, 9 ⟨ab⟩, 26 ⟨aab⟩ -
     26 ⟨abb⟩, 60 ⟨aaab⟩ - 255 ⟨aabb⟩ + 60 ⟨abbb⟩,
    119 ⟨aaaab⟩ - 1504 ⟨aaabb⟩ + 118 ⟨aabab⟩ +
     1504 ⟨aabbb⟩ + 1386 ⟨ababb⟩ - 119 ⟨abbbb⟩]]
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The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.
• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.