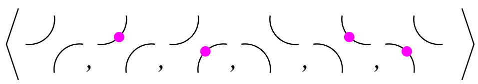
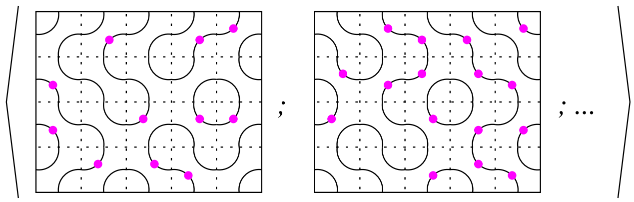
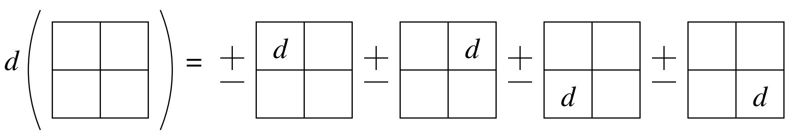
I don't Understand Khovanov–Rozansky Homology

Local state spaces:

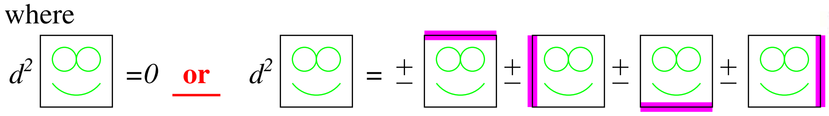
$V=$ 

$V^{\otimes(4\times 5)} =$ 

Local differentials:

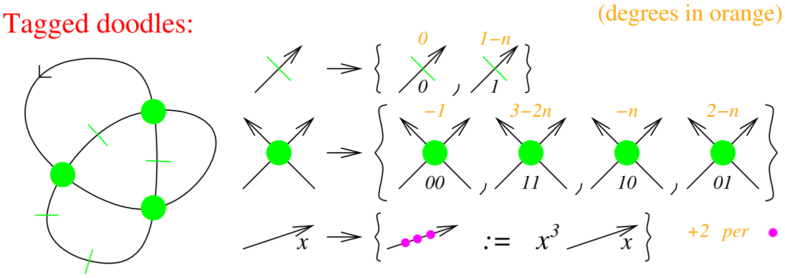


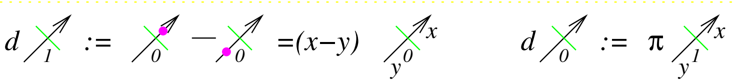
where



Tagged doodles:

(degrees in orange)

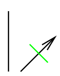





```
In[1]:= n = 2; π_{i_,j_} := Cancel[(x_i^{n+1} - x_j^{n+1})/(x_i - x_j)];  π_{1,2}
Out[1]= x1^2 + x1 x2 + x2^2
```

$$\text{In[2]:= } L = \begin{pmatrix} 0 & x_1 - x_2 \\ \pi_{1,2} & 0 \end{pmatrix};$$

```
Expand[L.L] // MatrixForm
```

} Set $L=d$ 

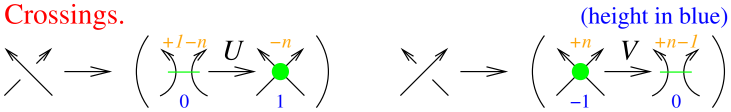
(deg $d = n+1$)

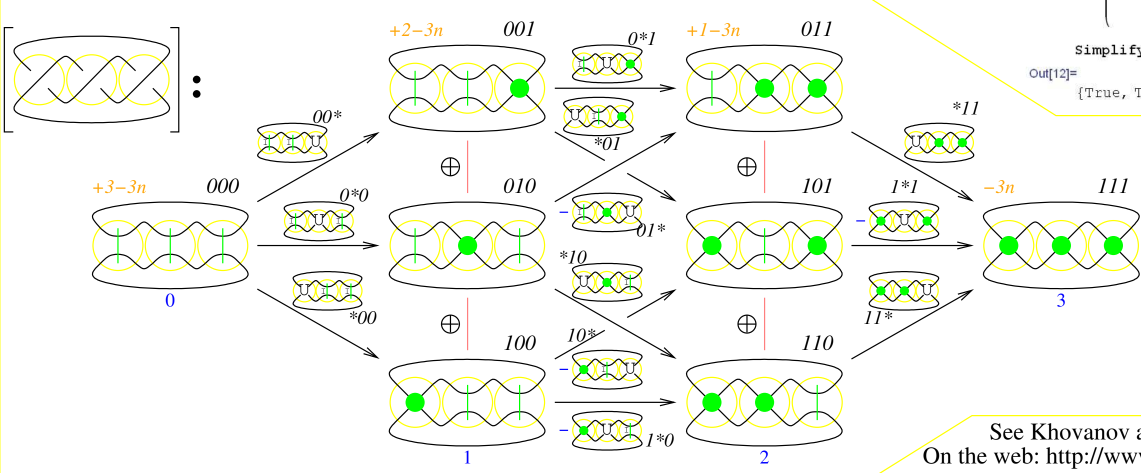
Out[3]//MatrixForm=

$$\begin{pmatrix} x_1^3 - x_2^3 & 0 \\ 0 & x_1^3 - x_2^3 \end{pmatrix}$$

Crossings.

(height in blue)





Matrix factorizations:

$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \qquad AB = BA = \omega I$$

$$\begin{tikzcd} M^0 \arrow[r, "A"] \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] & M^1 \arrow[r, "B"] \arrow[d, shift right, "U^1"'] \arrow[d, shift left, "V^1"] \arrow[dl, "h^1"'] & M^0 \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] \arrow[dl, "h^0"'] \\ N^0 \arrow[r, "A'"] & N^1 \arrow[r, "B'"] & N^0 \end{tikzcd}$$

A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

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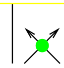
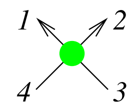
Likewise, set $Q=d$  with:

$$\text{In[4]:= } Q := \begin{pmatrix} 0 & 0 & v_1 & v_2 \\ 0 & 0 & u_2 & -u_1 \\ u_1 & v_2 & 0 & 0 \\ u_2 & -v_1 & 0 & 0 \end{pmatrix};$$

```
{v1, v2} = {x1 + x2 - x3 - x4, x1 x2 - x3 x4};
```



(deg $d = n+1$)

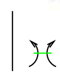
$$\text{In[6]:= } g[s\_, p\_] := s^{n+1} + (n+1)\sum_{i=1}^{(n+1)/2} \frac{(-1)^i}{i}\,\text{Binomial}[n-i,\ i-1]\, s^{n+1-2i} p^i;$$

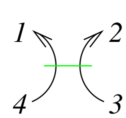
```
g[x + y, x y] // Expand
Out[6]= x^3 + y^3
```

$$\text{In[7]:= } \{u_1, u_2\} = \text{Cancel}\left[\left\{\frac{g[x_1+x_2,\ x_1 x_2] - g[x_3+x_4,\ x_1 x_2]}{v_1},\ \frac{g[x_3+x_4,\ x_1 x_2] - g[x_3+x_4,\ x_3 x_4]}{v_2}\right\}\right]$$

$$\text{Out[7]= } \{x_1^2 - x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_1 x_4 + x_2 x_4 + 2 x_3 x_4 + x_4^2,\ -3\,(x_3 + x_4)\}$$

```
In[8]:= ω = u1 v1 + u2 v2 // Expand
Out[8]= x1^3 + x2^3 - x3^3 - x4^3
In[9]:= Simplify[Q.Q == ω IdentityMatrix[4]]
Out[9]= True
```

Example: Set $P=d$ 



In[10]:=

$$P = \begin{pmatrix} 0 & 0 & x_1 - x_4 & x_2 - x_3 \\ 0 & 0 & \pi_{2,3} & -\pi_{1,4} \\ \pi_{1,4} & x_2 - x_3 & 0 & 0 \\ \pi_{2,3} & x_4 - x_1 & 0 & 0 \end{pmatrix};$$

```
In[11]:=
Simplify[P.P == ω IdentityMatrix[4]]
Out[11]=
True
```

In[12]:=

$$U = \begin{pmatrix} x_4 - x_2 & 0 & 0 & 0 \\ \frac{u_1 + x_4 u_2 - \pi_{2,3}}{x_1 - x_4} & 1 & 0 & 0 \\ 0 & 0 & x_4 & -x_2 \\ 0 & 0 & -1 & 1 \end{pmatrix}; \quad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u_1 + x_1 u_2 - \pi_{2,3}}{x_4 - x_1} & x_1 - x_3 & 0 & 0 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 1 & x_1 \end{pmatrix};$$

```
Simplify[{U.P == Q.U, V.Q == P.V}]
Out[12]=
{True, True}
```

Why am I happy?

1. The ugly formulas for L, Q, U, V; from where they come?
2. Where is the relationship with gl(n), representations and intertwiners?
3. Can you take the Euler characteristic before taking homology?
4. Is this computable?

See Khovanov and Rozansky, arXiv:math.QA/0401268, 0505056