# Nantel's Question, Categorification, Trace Groups

Dror Bar–Natan at York University, November 2004, http://www.math.toronto.edu/~drorbn/Talks/York–041108/

Dear Nantel, ~July 28, 2004

Let $p_n = \sum x_i^n$ denote the the $n$th power sum symmetric polynomial in the variables $\underline{x} = (x_i)_{i=1}^k$ and let $e_j$ denote the elementary symmetric polynomial in these variables (so $\prod_{i=1}^k (1 + x_i t) = \sum_{j=0}^k e_j(\underline{x}) t^j$). Write $p_n$ as a polynomial $F = F_{n,k}$ in the variables $\underline{e} = (e_j)$, so $p_n = F(\underline{e})$. What can you say about the Jacobian algebra $\Xi_{n,k}$ of $F$ — the algebra of polynomials in the variables $\underline{e}$ modulo all the first partial derivatives of $F$,

$$\Xi_{n,k} := \mathbb{Q}[e_1, \dots, e_k] \Big/ \left\langle \frac{\partial F}{\partial e_1}, \dots, \frac{\partial F}{\partial e_k} \right\rangle ?$$

In particular, is it true that $\dim \Xi = \binom{n-1}{k}$?

Sincerely,

Dror Bar-Natan (and Dylan Thurston).

## Conjecture:
(I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex–level.

I. Frenkel

## Quantum algebra:

Claim. If $ba = qab$ then

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}$$

where

$$(n)_q := 1 + q + \dots + q^{n-1},$$

$$(n)!_q := (1)_q (2)_q \cdots (n)_q,$$

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}.$$

## The Jones polynomial:

$$\hat{J} : \overcrossing \mapsto q\,)(\, - q^2 \asymp,$$

$$\hat{J} : \undercrossing \mapsto -q^{-2} \asymp + q^{-1}\,)(,$$

$$\bigcirc = q + q^{-1}$$

V. Jones

## Traces, dimensions, Lefschetz and Euler:

$$\tau(FG) = \tau(GF) \qquad \dim_\tau \mathcal{O} := \tau(I_\mathcal{O})$$

$$\tau(F) := \sum_r (-1)^r \tau(F^r) \qquad \chi_\tau(\Omega) := \tau(I_\Omega)$$

S. Lefschetz

## Homotopy invariance:

$$F - G = hd + dh \implies$$

L. Euler

$$\begin{aligned}\tau(F) - \tau(G) &= \sum_r (-1)^r \tau(F^r - G^r) \\ &= \sum_r (-1)^r \tau(h^{r+1}d^r + d^{r-1}h^r) \\ &= \sum_r (-1)^r \tau(h^{r+1}d^r - d^r h^{r+1}) = 0,\end{aligned}$$

$$GF \sim I_{\Omega_a},\ FG \sim I_{\Omega_b} \implies$$

$$\chi_\tau(\Omega_a) = \tau(I_{\Omega_a}) = \tau(GF) = \tau(FG) = \tau(I_{\Omega_b}) = \chi_\tau(\Omega_b)$$

## Complexes:

$$\Omega = \left( \Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \longrightarrow \Omega^{n_+} \right)$$

## Morphisms:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow \cdots \\ & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & \\ \cdots \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow \cdots \end{array}$$

## Homotopies:

$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ F^{r-1} \downdownarrows G^{r-1} & \swarrow h^r & F^r \downdownarrows G^r & \swarrow h^{r+1} & F^{r+1} \downdownarrows G^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

## The trace group:

$$\Xi(\mathcal{C}) := \bigoplus_{\mathcal{O} \in \mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{O}, \mathcal{O}) \Big/$$ the trace relation: $FG = GF$ whenever $F : \mathcal{O}_1 \to \mathcal{O}_2$ and $G : \mathcal{O}_2 \to \mathcal{O}_1$.

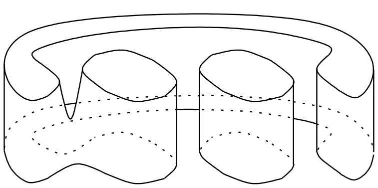
For matrices,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

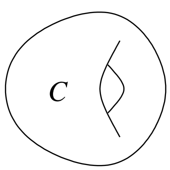
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1)$$

(Thanks, Dylan)

## Cobordisms:




$\Xi$: Surfaces in a solid torus




See also http://www.math.toronto.edu/~drorbn/papers/Cobordism