Math 157 Analysis I — Sample Final Exam

University of Toronto, April 16, 2003
web version: http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Final/Sample.html
last yers's exam: http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Final/LastYear.pdf

Solve the following 6 problems. Each is worth 20 points although they may have unequal difficulty, so the maximal possible total grade is 120 points. Write your answers in the space below the problems and on the front sides of the extra pages; use the back of the pages for scratch paper. Only work appearing on the front side of pages will be graded. Write your name and student number on each page. If you need more paper please ask the presiding officers.

Duration. You have 3 hours to write this exam.

Allowed Material: Any calculating device that is not capable of displaying text or graphs.

Good Luck!

Problem 1. We say that a set A of real numbers is *dense* if for any open interval I, the intersection $A \cap I$ is non-empty.

1. Give an example of a dense set A whose complement $A^c = \{x \in \mathbb{R} : x \notin A\}$ is also dense.
2. Give an example of a non-dense set B whose complement $B^c = \{x \in \mathbb{R} : x \notin B\}$ is also not dense.
3. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x) = 0$ for every x in some dense set A, then $f(x) = 0$ for every $x \in \mathbb{R}$.

Problem 2. Sketch the graph of the function $y = f(x) = x^2e^x$. Make sure that your graph clearly indicates the following:

- The domain of definition of $f(x)$.
- The behaviour of $f(x)$ near the points where it is not defined (if any) and as $x \to \pm\infty$.
- The exact coordinates of the x- and y-intercepts and all minimas and maximas of $f(x)$.

Problem 3. Compute the following integrals:

1. $\displaystyle\int \frac{x^3+1}{x+2}dx$.
2. $\displaystyle\int \frac{x-1}{4x^2+1}dx$.
3. $\displaystyle\int e^{ax}\sin bx\,dx \ \ (a, b \in \mathbb{R})$.

4. $\displaystyle\int x \log \sqrt{1+x^2}\, dx$.

5. $\displaystyle\int_0^1 \sqrt{1-x^2}\, dx$.

Problem 4. Agents of the CSIS have secretly developed two functions, $c(x)$ and $s(x)$, that have the following properties:

- $s(x+y) = s(x)c(y) + c(x)s(y)$ and $c(x+y) = c(x)c(y) - s(x)s(y)$ for all $x, y \in \mathbb{R}$.
- $c(0) = 1$ and $s(0) = 0$.
- $c'(0) = 0$ and $s'(0) = 1$.

Prove the following:

1. c and s are everywhere differentiable and $c' = -s$ and $s' = c$.
2. $c'' = -c$ and $s'' = -s$.
3. $c(x) = \cos x$ and $s(x) = \sin x$ for all $x \in \mathbb{R}$.

Problem 5.

1. Prove that if a sequence (f_n) of integrable functions on an interval $[a, b]$ converges uniformly on that interval to a function f, then the function f is integrable on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.
2. Prove that the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$ converges uniformly to some function $f(x)$ on $[-1, 1]$ and write a series of numbers whose sum is $\int_{-1}^{1} f(x)dx$.

Problem 6. Let $(z_n = x_n + iy_n)$ be a sequence of complex numbers and let $z = x + iy$ be another complex number.

1. Prove that the sequence (z_n) is bounded iff the sequences (x_n) and (y_n) are both bounded.
2. Prove that $\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.
3. Prove that if the sequence (z_n) is bounded then it has a convergent subsequence.