Math 157 Analysis I — Solution of the Final Exam

web version: http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Final/Solution.html

Problem 1. Let f and g denote functions defined on some set A.

1. Prove that
$$\sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

2. Find an example for a pair f, g for which
$$\sup_{x\in A}(f(x)+g(x)) = \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

3. Find an example for a pair f, g for which
$$\sup_{x\in A}(f(x)+g(x)) < \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

Solution.

1. For any $x \in A$, $f(x) \le \sup_{x\in A} f(x)$ and $g(x) \le \sup_{x\in A} g(x)$ and hence $f(x)+g(x) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$. Thus $\sup_{x\in A} f(x) + \sup_{x\in A} g(x)$ is an upper bound for $f(x)+g(x)$ on A, and hence it is no smaller than the least upper bound for $f(x)+g(x)$ on A, which is $\sup_{x\in A}(f(x)+g(x))$.

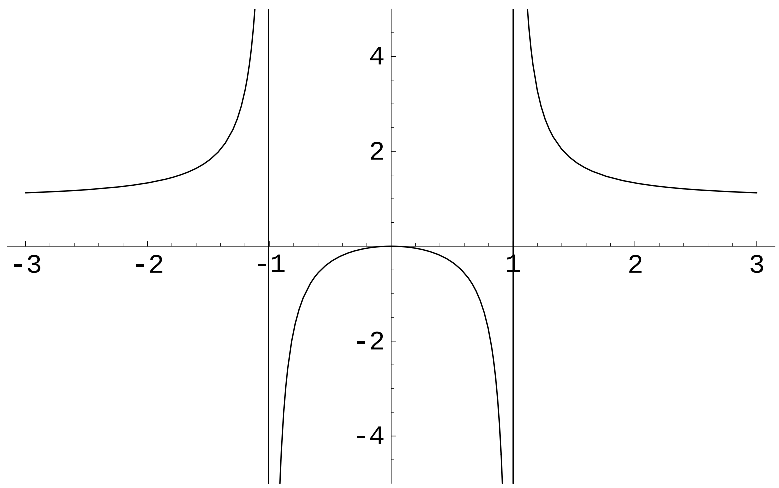
2. Take say f and g to be the constant functions 0, and then $\sup_{x\in A}(f(x)+g(x))$ and $\sup_{x\in A} f(x) + \sup_{x\in A} g(x)$ are both 0.

3. Take say $f(x) = x$ and $g(x) = -x$ on $A = [0,1]$. Then $f(x)+g(x) = 0$ and hence $\sup_{x\in A}(f(x)+g(x)) = 0$ while $\sup_{x\in A} f(x) = 1$ and $\sup_{x\in A} g(x) = 0$ and hence $\sup_{x\in A} f(x) + \sup_{x\in A} g(x) = 1$. Thus $\sup_{x\in A}(f(x)+g(x)) = 0 < 1 = \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$ as required.

Problem 2. Sketch the graph of the function $y = f(x) = \dfrac{x^2}{x^2-1}$. Make sure that your graph clearly indicates the following:

- The domain of definition of $f(x)$.
- The behaviour of $f(x)$ near the points where it is not defined (if any) and as $x \to \pm\infty$.
- The exact coordinates of the x- and y-intercepts and all minimas and maximas of $f(x)$.

Solution. $f(x)$ is defined for $x \neq \pm 1$, and the following limits are easily computed: $\lim_{x\to\pm\infty} f(x) = 1$, $\lim_{x\to -1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = -\infty$. The only solution for $f(x) = 0$ is $x = 0$, hence the only intersection of the graph of $f(x)$ with the axes is at $(0,0)$. Other than at $x = 0$, the numerator of f is always positive, hence the sign of the function is determined by the sign of the denominator $x^2 - 1$. Thus $f(x) \leq 0$ for $|x| < 1$ and $f(x) > 0$ for $|x| > 1$. Finally $f'(x) = \frac{2x(x^2-1)-x^2 2x}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}$ and thus f' is positive and f is increasing (locally) for $x < 0$ and f' is negative and f is decreasing (locally) for $x > 0$. Thus overall the graph is:



Problem 3. Compute the following integrals:

1. $\displaystyle\int \frac{x^2+1}{x+1}dx$

   Solution. By long division of polynomials, $x^2 + 1 = (x+1)(x-1) + 2$. Thus we can rewrite our integral as a sum of two terms as follows

$$\int \frac{(x+1)(x-1)}{x+1}dx + \int \frac{2}{x+1}dx = \int (x-1)dx + 2\int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} - x + 2\log(x+1).$$

2. $\displaystyle\int \frac{x+1}{x^2+1}dx$

   Solution. Again we rewrite the integral as a sum of two terms. On the first we perform the substitution $u = x^2$; the second is elementary:

$$\frac{1}{2}\int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u+1} + \arctan x = \frac{1}{2}\log(u+1) + \arctan x$$

$$= \frac{1}{2}\log(x^2+1) + \arctan x.$$

3. $\int x^2 \sin x \, dx$

Solution. We integrate by parts twice, as follows:

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int 2x(-\cos x)dx$$

$$= -x^2 \cos x - 2x(-\sin x) - \int 2(-\sin x)dx = 2x \sin x - x^2 \cos x + 2\cos x.$$

4. $\int \frac{dx}{\sqrt{1+e^x}}$

Solution. Set $u = \sqrt{1+e^x}$ and then $e^x = u^2 - 1$ and $du = \frac{e^x dx}{2\sqrt{1+e^x}} = \frac{(u^2-1)dx}{2u}$ and so $dx = \frac{2udu}{u^2-1}$ and

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{2udu}{u(u^2-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1}$$

$$= \log(u-1) - \log(u+1) = \log \frac{u-1}{u+1} = \log \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}.$$

5. $\int_0^\infty e^{-x} dx$

Solution.

$$\int_0^\infty e^{-x} dx = \lim_{X\to\infty} - e^{-x}\Big|_0^X = \lim_{X\to\infty} e^{-0} - e^{-X} = 1.$$

Or using a shorter and less precise notation, but good enough —

$$\int_0^\infty e^{-x} dx = - e^{-x}\Big|_0^\infty = e^{-0} - e^{-\infty} = 1.$$

Problem 4. Agents of the CSIS have secretly developed a function $e(x)$ that has the following properties:

- $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.
- $e(0) = 1$
- e is differentiable at 0 and $e'(0) = 1$.

Prove the following:

1. e is everywhere differentiable and $e' = e$.
2. $e(x) = e^x$ for all $x \in \mathbb{R}$. The only lemma you may assume is that if a function f satisfies $f'(x) = 0$ for all x then f is a constant function.

Solution.

1. The given fact that $1 = e'(0)$ means that $1 = \lim_{h\to 0} \frac{e(h)-e(0)}{h} = \lim_{h\to 0} \frac{e(h)-1}{h}$. Hence, using $e(x+h) = e(x)e(h)$ we get

$$\lim_{h\to 0} \frac{e(x+h)-e(x)}{h} = \lim_{h\to 0} \frac{e(x)e(h)-e(x)}{h} = e(x) \lim_{h\to 0} \frac{e(h)-1}{h} = e(x).$$

   This proves both that e is differentiable at x and that $e'(x) = e(x)$.

2. Consider $q(x) = e(x)e^{-x}$. Differentiating we get

$$q'(x) = e'(x)e^{-x} + e(x)(e^{-x})' = e(x)e^{-x} - e(x)e^{-x} = 0.$$

   Hence $q(x)$ is a constant function. But $q(0) = e(0)e^0 = 1 \cdot 1 = 1$, hence this constant must be 1. So $e(x)e^{-x} = 1$ and thus $e(x) = e^x$.

Problem 5.

1. Prove that if a sequence of continuous functions f_n converges uniformly to a function f on some interval $[a, b]$, then f is continuous on $[a, b]$.
2. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$ converges on $(-\infty, \infty)$ and that its sum is a continuous function of x.

Solution.

1. See Spivak's Theorem 2 of Chapter 24.
2. $|\frac{1}{2^n} \sin(3^n x)| \le \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges. Hence by the Weierstrass M-Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$ converges uniformly. As each of the terms $\frac{1}{2^n} \sin(3^n x)$ is continuous, the first part of this question implies that so is the sum.

Problem 6. Prove that the complex function $z \mapsto \bar{z}$ is everywhere continuous but nowhere differentiable.

Solution. The key point is that $|w| = |\bar{w}|$ for every complex number w. Let $\epsilon > 0$ and set $\delta = \epsilon$. Now if $|z - z_0| < \delta$ then $|\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon$. This proves the continuity of $z \mapsto \bar{z}$. Let us check if this function is differentiable:

$$\lim_{h\to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h\to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}.$$

If we restrict our attention to real h then the latter quotient is always 1, so the limit would be 1. If we restrict our attention to imaginary h, $h = iy$ with real y, then that quotient is $\frac{\bar{h}}{h} = \frac{-iy}{iy} = -1$ so the limit would be -1. Hence the limit cannot exist and $z \mapsto \bar{z}$ is not differentiable at (an arbitrary) z.

The results. 76 students took the exam; the average grade was 72.66/120, the median was 71.5/120 and the standard deviation was 25.5. The overall grade average for the course (of $X = 0.05T_1 + 0.15T_2 + 0.1T_3 + 0.1T_4 + 0.2HW + 0.4 \cdot 100(F/120)$) was 66.92, the median was 64.9 and the standard deviation was 17.16. Finally, the transformation $X \mapsto 100(X/100)^\gamma$ was applied to the grades, with $\gamma = 0.82$. This made the average grade 71.55, the median 70 and the standard deviation 15.31. There were 25 A's (grades above 80) and 5 failures (grades below 50).