Series

web version: http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/Series/Series.html

Definition. $\sum_{n=1}^{\infty} a_n$ is "convergent" or "summable" and $\sum_{n=1}^{\infty} a_n = s$ if $\sum_{n=1}^{N} a_n \to s$. Otherwise $\sum_{n=1}^{\infty} a_n$ is "divergent".

Claim. When the right hand sides exist, so do the left hand sides and the equalities hold:

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n.$$

Claim. (The Boundedness Criterion) A nonnegative sequence is summable iff its partial sums are bounded.

Claim. (Cauchy's Criterion) (a_n) is summable iff

$$\lim_{n,m\to\infty} a_n + \cdots + a_m = 0.$$

Claim. (The Vanishing Condition) If (a_n) is summable then $a_n \to 0$.

Theorem 1. (The Comparison Test) If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then so does $\sum a_n$.

Theorem 2. If $a_n > 0$ and $b_n > 0$ and $\lim a_n/b_n = c \neq 0$ then $\sum a_n$ converges iff $\sum b_n$ converges.

Theorem 3. (The Ratio Test) If $a_n > 0$ and $\lim a_{n+1}/a_n = r$, then $\sum a_n$ converges if $r < 1$ and diverges if $r > 1$.

Theorem 4. (The Integral Test) If $f > 0$ and f is decreasing on $[1, \infty)$ and $f(n) = a_n$, then $\sum a_n$ converges iff $\int_1^\infty f$ converges.

Definition. $\sum_{n=1}^{\infty} a_n$ is "absolutely convergent" if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 5. An absolutely convergent series is convergent. A series is absolutely convergent iff the series formed from its positive terms and its negative terms are both convergent.

Theorem 6. (Leibnitz's Theorem) If a_n is non-increasing and $\lim a_n = 0$ then $\sum(-1)^n a_n$ converges.

Theorem 7. (skipped) If $\sum a_n$ converges but not absolutely, then for any number s there is a rearrangement (b_n) of (a_n) so that $s = \sum b_n$.

Theorem 8. If $\sum a_n$ converges absolutely and (b_n) is a rearrangement of (a_n), then $\sum b_n$ also converges absolutely and $\sum a_n = \sum b_n$.

Theorem 9. If $\sum a_n$ and $\sum b_n$ converge absolutely and the sequence c_n is composed of all the products of the form $a_i b_j$, then $\sum c_n = \sum a_n \cdot \sum b_n$.