University of Toronto

Department of Mathematics

Math 157 Exam 1

Monday, October 22, 2001

One hour and fifty minutes

There are five problems, each worth 20 points although they do not have equal difficulty. Write your answer in the space below the problem; use the back of the sheets and the last page for scratch paper. Only work appearing on the front of the page will be graded. Write your name on each page.

No calculators or other devices.

For Grading Use

1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

1. (a) State the definitions of $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} f(x) = -\infty$.

   (b) For each of the following statements, give either

   - a proof, if the conclusion is true for all functions f and all points $a \in \mathbb{R}$ that satisfy the hypotheses, or
   - a counterexample: a specific function f and a specific point a for which the hypothesis is true but the conclusion is false.

   i. If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \dfrac{1}{f(x)} = \infty$.

   ii. If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \dfrac{1}{f(x)} = 0$.

   iii. If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \dfrac{1}{f(x)^2} = \infty$.

   iv. If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \dfrac{1}{f(x)^2} = 0$.

2. Let $f(x) = \cos(1/x)$ for $x > 0$.

   (a) For what values of x is $f(x) = 1$? For what x is $f(x) = -1$? (*Hint:* $\cos(x) = \pm 1$ where $\sin(x) = 0$.)

   (b) If f continuous on its domain? Does $\lim_{x \to 0} f(x)$ exist?

   (c) Prove the following statement:

   > For every c with $-1 \leq c \leq 1$, and for every $\delta > 0$, there is at least one x with $0 < x < \delta$ for which $f(x) = c$.

3. Evaluate the following limits if they exist. You do not have to prove your result, but show your work and say what theorems you rely on.

   (a) $\lim_{x \to 1} \dfrac{x^n - 1}{x - 1}$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

   (b) $\lim_{x \to 2} \sqrt{5 + \sqrt[3]{2x^5}}$ (you may assume that $\sqrt{\cdot}$ and $\sqrt[3]{\cdot}$ are continuous)

   (c) $\lim_{x \to 0} \dfrac{\sqrt{1+x}}{x}$

   (d) $\lim_{x \to 0} \dfrac{\sqrt{1-x}}{x}$

   (e) $\lim_{x \to 0} \left(\dfrac{\sqrt{1+x}}{x} - \dfrac{\sqrt{1-x}}{x} \right)$

4. (a) Prove by induction that $(1+x)^n \geq 1+nx$ for $x > -1$ and $n \in \mathbb{N}$.

   (b) More generally, suppose that $x_1, \ldots, x_n$ are all > -1, and that they are either all ≥ 0 or all ≤ 0. Prove by induction that

$$(1+x_1)\cdots(1+x_n) \;\geq\; 1+x_1+\cdots+x_n.$$

5. Sketch the graphs of each of the following functions. For what values of x is each continuous? The domain of definition is all $x \in \mathbb{R}$ unless otherwise specified.

   (a) $f(x) = \dfrac{x + |x|}{2}$

   (b) $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0. \end{cases}$ If it is not continuous, can it be made continuous by choosing a different value for $f(0)$?

   (c) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & \text{for } x \neq 1 \\ 0, & \text{for } x = 1 \end{cases}$ If it is not continuous, can it be made continuous by choosing a different value for $f(1)$?

   (d) $f(x) = \begin{cases} \dfrac{x^2 - 2}{x - 2}, & \text{for } x \neq 2 \\ 0, & \text{for } x = 2 \end{cases}$ If it is not continuous, can it be made continuous by choosing a different value for $f(2)$?

   (e) $f(x) = \begin{cases} \sin \dfrac{1}{x}, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ for $x > 0$.