Dror Bar-Natan: Classes: 2002-03: Math 157 - Analysis I:

Sample Term Exam 1

web version:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/TermExam1/SampleExam.html
last year's exam:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/TermExam1/LastYear.pdf

Solve the following 5 problems. Each is worth 20 points although they may have unequal difficulty. Write your answers in the space below the problems and on the front sides of the extra pages; use the back of the pages for scratch paper. Only work appearing on the front side of pages will be graded. Write your name and student number on each page. If you need more paper please ask the tutors. You have an hour and 50 minutes.
Allowed Material: Any calculating device that is not capable of displaying text.

Good Luck!

(In the real exam each of the questions will appear on a separate page and there will be several blank pages stapled with your exam booklet. You will have space for your name and student number at the top of each page.)

Problem 1.

1. Prove directly from the postulates for the real numbers and from the relevant definitions that if $a < b$ and $c < 0$ then $ac > bc$.

2. Use induction to prove that any integer n can be written in exactly one of the following three forms: $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, where k is also an integer.

3. Prove that there is no rational number r such that $r^2 = 3$.

Problem 2. We say that a function g is an *inverse* of a function f if $f \circ g = g \circ f = I$, where I is the *identity function,* defined by $I(x) = x$ for all x. Show that a function f has an inverse g if and only if the following two conditions are satisfied:

1. If $x \neq y$ then $f(x) \neq f(y)$.

2. For every a there is an x such that $a = f(x)$.

Problem 3. Sketch, to the best of your understanding, the graph of the function

$$f(x) = x + \frac{1}{x}.$$

(What happens for x near 0? For large x? Where does the graph lie relative to the graph of the identity function?)

Problem 4. Suppose that A_n is, for each natural number n, some *finite* set of numbers and that A_n and A_m have no members in common if $n \neq m$. Define f as follows:

$$f(x) = \begin{cases} 1/n, & \text{if } x \in A_n \text{ for some } n \\ 0, & \text{if } x \notin A_n \text{ for all } n. \end{cases}$$

Prove that $\lim_{x \to a} f(x) = 0$ for all a.

Problem 5. Suppose that f satisfies $f(x + y) = f(x) + f(y)$ for all x and y and that f is continuous at 0. Prove that f is continuous everywhere.