Math 157 Analysis I — Solution of Term Exam 1

web version:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/TermExam1/Solution.html

Problem 1.

1. Prove directly from the postulates for the real numbers and from the relevant definitions that if $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. If you plan to use a formula such as $b^2 - a^2 = (b-a)(b+a)$ you don't need to prove it, but of course you have to be very clear about how it is used.

2. Use induction to prove that any integer n can be written in exactly one of the following two forms: $n = 2k$ or $n = 2k+1$, where k is also an integer.

3. Prove that there is no rational number r such that $r^3 = 2$.

Solution.

1. If $a = 0$ then $a^2 < b^2$ means that $0 < b^2$ and therefore $b \neq 0$. Along with $a \geq 0$, $b \geq 0$ and P11, it follows that $b + a > 0$. If $a > 0$ then again along with $b \geq 0$ and P11 we get that $b + a > 0$, so in either case $b + a > 0$ is assured. Now $a^2 < b^2$ is by definition the same as $b^2 - a^2 > 0$, and therefore

$$(b-a)(b+a) > 0. \tag{1}$$

   Had $(b-a)$ been negative, then $-(b-a)$ would have been positive and by $b + a > 0$ and P12 we'd have that $-(b-a)(b+a) > 0$, contradicting Equation (1). Hence $(b-a)$ is positive, and this by definition means that $b > a$.

2. First we show using induction that every natural number (positive integer) n can be written in at least one of the forms $n = 2k$ or $n = 2k+1$ for an integer k. Indeed, for $n = 1$ we write $1 = 2 \cdot 0 + 1$ as required. Now if n is of the form $2k$ for an integer k, then $n + 1 = 2k + 1$ is of the second allowed form, and if n is of the form $2k+1$ for some integer k then $n + 1 = 2k + 1 + 1 = 2(k+1)$ is of the first allowed form, for $k+1$ is also an integer. Therefore if n can be written in either of the required forms then so is $n + 1$, and the inductive proof is completed.

   Let's deal with 0 and with the negative integers now. First, $0 = 2 \cdot 0$ so 0 is of the form $2k$. Next, if $n < 0$ is an integer, then $(-n)$ is a positive integer and therefore $(-n) = 2k$ or $(-n) = 2k+1$. In the former case, $n = 2(-k)$ and we are done. In the latter case, $n = 2(-k) - 1 = 2(-k-1) + 1$ and again we are done.

   Finally, an integer cannot be of both forms at the same time, for if we could write $n = 2k_1$ and $n = 2k_2 + 1$ with integer k_1 and k_2, then we'd have that $2k_1 = 2k_2 + 1$, which is $2(k_1 - k_2) = 1$. But $k_1 - k_2$ is an integer and it is easy to show that 1 is not twice an integer.

   Having said all that, we can call the integers of the form $2k$ "even" and the integers of the form $2k+1$ "odd", and then every integer is either even or odd but never both.

3. First, $(2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ and hence if n is odd then so is n^3. Therefore if n^3 is even then so is n. Now assume by contradiction that there is some rational number r with $r^3 = 2$ and write $r = p/q$ where p and q are integers and q is the least positive integer for which it is possible to present r in this form. Now $r^3 = p^3/q^3 = 2$ hence $p^3 = 2q^3$ is even hence p is even hence we can find an integer k with $p = 2k$. But then $p^3 = 2q^3$ becomes $8k^3 = 2q^3$ and hence $q^3 = 2(2k^3)$ so q^3 is even and hence so is q, so $q = 2l$ for some positive l (which of course is smaller than q itself). But now $r = p/q = 2k/2l = k/l$ contradicting the minimality of q. Therefore there is no rational number r with $r^3 = 2$.

Problem 2.

1. Suppose $f(x) = x+1$. Are there any functions g such that $f \circ g = g \circ f$?
2. Suppose that f is a constant function. For which functions g does $f \circ g = g \circ f$?
3. Suppose that $f \circ g = g \circ f$ for *all* functions f. Show that g is the identity function $g(x) = x$.

Solution.

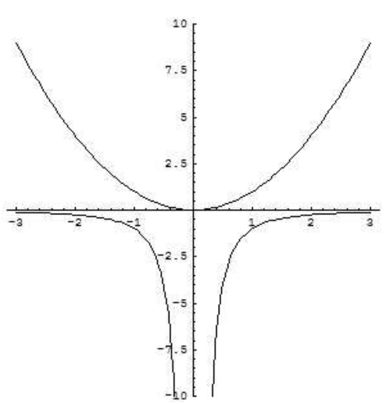
1. Yes. For example, the identity function $g(x) = x$ has this property.
2. Suppose $f(x) = c$ for all x. Then $f \circ g = g \circ f$ if and only if $\forall x\, (f \circ g)(x) = (g \circ f)(x)$ iff $\forall x\, f(g(x)) = g(f(x))$ iff $\forall x\, c = g(c)$ iff $c = g(c)$. So g satisfies $f \circ g = g \circ f$ iff $g(c) = c$.
3. If $f \circ g = g \circ f$ for *all* functions f then in particular $f \circ g = g \circ f$ for all constant functions $f(x) = c$. But then by the previous part for all constant c we have $g(c) = c$ and this precisely means that g is the identity function $g(x) = x$.

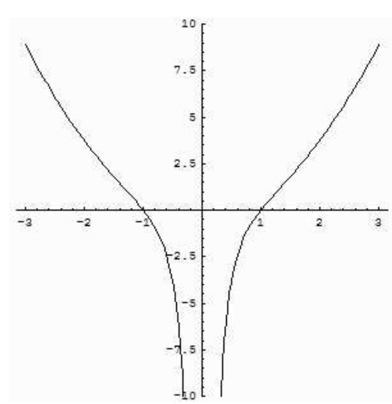
Problem 3. Sketch, to the best of your understanding, the graph of the function

$$f(x) = x^2 - \frac{1}{x^2}.$$

(What happens for x near 0? For large x? Where does the graph lie relative to the graph of the function $y = x^2$?)

Solution. The first graph below shows x^2 (above the x axis) and $-1/x^2$ (below the x axis. The second shows the sum of the two, the desired function $x^2 - 1/x^2$:





For x near 0 our function goes to $-\infty$, for large x it goes to $+\infty$. It is always below x^2 but for large x it is very near x^2.

Problem 4. Write the definition of $\lim_{x\to a} f(x) = l$ and give examples to show that the following definitions of $\lim_{x\to a} f(x) = l$ do not agree with the standard one:

1. For all $\delta > 0$ there is an $\epsilon > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - l| < \epsilon$.

2. For all $\epsilon > 0$ there is a $\delta > 0$ such that if $|f(x) - l| < \epsilon$, then $0 < |x - a| < \delta$.

Solution. The definition is: For every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| < \delta$ we have that $|f(x) - l| < \epsilon$. The required examples:

1. This is satisfied whenever $|f|$ is bounded and regardless of its limit. Indeed, choose ϵ bigger than $|l| + M$ where M is a bound on $|f|$, and $|f(x) - l| < \epsilon$ is always true.

2. According to this definition, for example, $\lim_{x\to a} c = c$ is false, and hence it cannot be equivalent to the standard definition. Indeed, in this case $|f(x) - l| < \epsilon$ means $0 = |c - c| < \epsilon$. This imposes no condition on x, so $|x - a|$ need not be smaller than δ.

Problem 5. Suppose that g is continuous at 0 and $g(0) = 0$ and that $|f(x)| \le \sqrt{|g(x)|}$ for all x. Show that f is continuous at 0.

Solution. At $x = 0$ the inequality $|f(x)| \le \sqrt{|g(x)|}$ reads $|f(0)| \le \sqrt{|g(0)|} = \sqrt{|0|} = 0$, and hence $f(0) = 0$. We claim that $\lim_{x\to 0} f(x) = 0$ and hence that f is continuous at 0. Indeed let $\epsilon > 0$ be given. Then $\epsilon^2 > 0$ and by the continuity of g at 0 we can find a $\delta > 0$ so that whenever $|x| < \delta$ we have that $|g(x)| < \epsilon^2$. But then if $|x| < \delta$ then $|f(x)| \le \sqrt{|g(x)|} = \sqrt{\epsilon^2} = \epsilon$ and the definition of $\lim_{x\to 0} f(x) = 0$ is satisfied.