Dror Bar-Natan: Classes: 2002-03: Math 157 - Analysis I:

Welcome Back!

armed with the known, we sail to explore the yet unknown

web version:
http://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/WelcomeBack/WelcomeBack.html

The Known:

Setting. f bounded on $[a,b]$, $P: a=t_0<t_1<\cdots<t_n=b$ a partition of $[a,b]$, $m_i=\inf_{[t_{i-1},t_i]} f(x)$, $M_i=\sup_{[t_{i-1},t_i]} f(x)$, $L(f,P)=\sum_{i=1}^n m_i(t_i-t_{i-1})$, $U(f,P)=\sum_{i=1}^n M_i(t_i-t_{i-1})$, $L(f)=\sup_P L(f,P)$, $U(f)=\inf_P U(f,P)$. Finally, if $U(f)=L(f)$ we say that "f is integrable on $[a,b]$" and set $\int_a^b f=\int_a^b f(x)dx=U(f)=L(f)$.

Theorem 13-1. For any two partitions $P_{1,2}$, $L(f,P_1)\leq U(f,P_2)$.

Theorem 13-2. f is integrable iff for every $\epsilon>0$ there is a partition P such that $U(f,P)-L(f,P)<\epsilon$.

Theorem 13-3. If f is continuous on $[a,b]$ then f is integrable on $[a,b]$.

Theorem 14-2. (The Second Fundamental Theorem of Calculus) If f' is integrable on $[a,b]$, where f is some differentiable function, then $\int_a^b f'=f(b)-f(a)$.

Theorem 13-4. If $a<c<b$ then $\int_a^b f=\int_a^c f+\int_c^b f$ (in particular, the rhs makes sense iff the lhs does).

The Yet Unknown:

Convention. $\int_a^a f:=0$ and if $b<a$ we set $\int_a^b f:=-\int_b^a f$.

Theorem 13-4'. $\int_a^b f=\int_a^c f+\int_c^b f$ so long as all integrals exist, no matter how a, b and c are ordered.

Theorem 13-5. If f and g are integrable on $[a,b]$ then so is $f+g$, and $\int_a^b f+g=\int_a^b f+\int_a^b g$.

Theorem 13-6. If f is integrable on $[a,b]$ and c is a constant, then cf is integrable on $[a,b]$ and $\int_a^b cf=c\int_a^b f$.

Theorem 13-7$^{\mathbf{a}}$. If $f\leq g$ on $[a,b]$ and both are integrable on $[a,b]$, then $\int_a^b f\leq\int_a^b g$.

Theorem 13-7. If $m\leq f(x)\leq M$ on $[a,b]$ and f is integrable on $[a,b]$ then $m(b-a)\leq\int_a^b f\leq M(b-a)$.

Theorem 13-8. If f is integrable on $[a,b]$ and F is defined on $[a,b]$ by $F(x)=\int_a^x f$, then F is continuous on $[a,b]$.

Theorem 14-1. (The First Fundamental Theorem of Calculus) Let f be integrable on $[a,b]$, and define F on $[a,b]$ by $F(x)=\int_a^x f$. If f is continuous at $c\in[a,b]$, then F is differentiable at c and $F'(c)=f(c)$.