Dror Bar-Natan: Classes: 2003-04: Math 1350F - Knot Theory:

The Final Exam

web version: http://www.math.toronto.edu/˜drorbn/classes/0304/KnotTheory/Final/Final.html

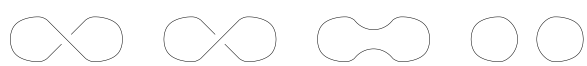
Solve and submit your solution of two (just two!) of the following three questions by noon on Tuesday January 6, 2004. Remember — **Elegance counts!!!** If you can type your solution, that's better. If you can't, at least copy it again to a clean sheet of paper. Formulas without words explaining them will not be accepted!

1. Prove in detail:
   (a) All torus knots, except for the obvious exceptions, are really knotted.
   (b) All knotted torus knots are prime.

2. The "Dubrovnik Polynomial" D (a variant of the "Kauffman Polynomial" L) is an invariant of framed links valued in rational functions in the variables a and z, satisfying the following relations:

$$\begin{aligned} D(\bigcirc) &= 1, && (1)\\ D(\text{⌒୪⌒}) &= aD(\frown\!\!\smile), && (2)\\ D(\text{⌒୪⌒}) &= a^{-1}D(\frown\!\!\smile), && (3)\\ D(\times) - D(\times) &= z(D(\asymp) - D()()). && (4)\end{aligned}$$

   (a) Compute $D(\bigcirc^k)$ (where $\bigcirc^k$ is the k-component unlink).
   Hint. One instance of relation (4) relates the following four knots; three of them are the unknot with different framings:



   (b) Prove that the above conditions determine D on all knots and links.
   (c) Set $z = e^{x/4} - e^{-x/4}$ and $a = \exp\left((N-1)\frac{x}{4}\right)$ and expand

$$D(K;\,z,a) = \sum_{m=0}^{\infty} D_m(K;\,N)x^m$$

   (here K stands for an arbitrary knot or link). Prove that for any m the coefficient D_m is a type m invariant of links with values in polynomials in N.
   (d) Determine the weight system of D_m and show that it is the weight system arising from the Lie algebra $so(N)$.

3. Claim: The integral operator given by the kernel

$$G_{ij}(x,y) = \frac{\epsilon_{ijk}}{4\pi}\frac{x^k - y^k}{|x-y|^3}$$

is an inverse of the differential operator $\star d$.

Explain what this claim means and prove it. This done, show that if $\gamma_{1,2}$ are disjoint space curves, then

$$\int dt_1 dt_2\, G_{ij}(\gamma_1(t_1), \gamma_2(t_2))\dot{\gamma}_1^i(t_1)\dot{\gamma}_2^j(t_2) = \int_{T^2} \Phi^\star \omega,$$

where $\Phi : T^2 \to S^2$ is the "direction of sight" map $\Phi(t_1, t_2) = \frac{\gamma_1(t_1)-\gamma_2(t_2)}{|\gamma_1(t_1)-\gamma_2(t_2)|}$ and where ω is the volume form of S^2 normalized so that the total volume of S^2 is 1.

Good Luck!!