Homework Assignment 6: Deframing

Assigned Thursday October 23; due Thursday October 30 in class.

Required reading. Sections 2 and 3 of my paper *On the Vassiliev Knot Invariants.*

Let $\Theta : \mathcal{A} \to \mathcal{A}$ be the multiplication operator by the chord diagram θ, and let $\partial_\theta = \frac{d}{d\theta}$ be the adjoint of multiplication by W_θ on $\mathcal{A}^\star$, where W_θ is the obvious dual of θ in $\mathcal{A}^\star$. Let $P : \mathcal{A} \to \mathcal{A}$ be defined by

$$P = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{n!} \partial_\theta^n.$$

The following assertions can be verified:

1. $[\partial_\theta, \Theta] = 1$, where $1 : \mathcal{A} \to \mathcal{A}$ is the identity map and where $[A, B] := AB - BA$ for any two operators.
2. P is a degree 0 operator; that is, $\deg Pa = \deg a$ for all $a \in \mathcal{A}$.
3. ∂_θ satisfies Leibnitz' law: $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$ for any $a, b \in \mathcal{A}$.
4. P is an algebra morphism: $P1 = 1$ and $P(ab) = (Pa)(Pb)$.
5. Θ satisfies the co-Leibnitz law: $\Box \circ \Theta = (\Theta \otimes 1 + 1 \otimes \Theta) \circ \Box$ (why does this deserve the name "the co-Leibnitz law"?).
6. P is a co-algebra morphism: $\eta \circ P = \eta$ (where η is the co-unit of $\mathcal{A}$) and $\Box \circ P = (P \otimes P) \circ \Box$.
7. $P\theta = 0$ and hence $P\langle\theta\rangle = 0$, where $\langle\theta\rangle$ is the ideal generated by θ in the algebra $\mathcal{A}$.
8. If $Q : \mathcal{A} \to \mathcal{A}$ is defined by
$$Q = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{(n+1)!} \partial_\theta^{(n+1)}$$
then $a = \theta Q a + P a$ for all $a \in \mathcal{A}$.
9. $\ker P = \langle\theta\rangle$.
10. P descends to a Hopf algebra morphism $\mathcal{A}^r \to \mathcal{A}$, and if $\pi : \mathcal{A} \to \mathcal{A}^r$ is the obvious projection, then $\pi \circ P$ is the identity of $\mathcal{A}^r$. (Recall that $\mathcal{A}^r = \mathcal{A}/\langle\theta\rangle$.)
11. $P^2 = P$.

To be handed in. Verify assertions 4, 5, 7 and 11 above.

Recommended for extra practice. Verify all the other assertions above.

Idea for a good deed. Prepare a beautiful TEX writeup (including the motivation and all the details) of the solution of this assignment for publication on the web. For all I know this information in this form is not available elsewhere.