UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

APRIL/MAY EXAMINATIONS 2005

Math 157Y1Y Analysis I — Final Exam

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Name: ______________________________ Student ID: __________________

Solve the following 6 problems. Each is worth 20 points although they may have unequal difficulty, so the maximal possible total grade is 120 points. Write your answers in the space below the problems and on the front sides of the extra pages; use the back of the pages for scratch paper. Only work appearing on the front side of pages will be graded. Write your name and student number on each page. If you need more paper please ask the presiding officers. This booklet has 12 pages.

Duration. You have 3 hours to write this exam.

Allowed Material: Any calculating device that is not capable of displaying text.

Good Luck!

For Grading Use Only

1	/20	4	/20
2	/20	5	/20
3	/20	6	/20
Total	/120		

web version: http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/Final/Exam.html

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Problem 1.

1. Define "the function f is continuous at a point l".
2. Define "$\lim_{x \to a} g(x) = l$".
3. Prove from these definitions that if f is continuous at l and if $\lim_{x \to a} g(x) = l$, then $\lim_{x \to a} f(g(x)) = f(l)$.
4. Find an example for funtions f and g defined over all of $\mathbb{R}$ and for which $\lim_{x \to a} g(x) = l$ and yet $\lim_{x \to a} f(g(x)) \neq f(l)$ (of course, f will not be continuous at l).

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Problem 2. Sketch the graph of the function $y = f(x) = x^2 e^{-x}$. Make sure that your graph clearly indicates the following:

- The domain of definition of f.
- The behaviour of f near the points where it is not defined (if any) and as $x \to \pm\infty$.
- The intervals over which f is increasing and the intervals over which f is decreasing.
- The exact coordinates of the x- and y-intercepts and all minimas and maximas of f.

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Problem 3. Compute the following derivative and the following integrals:

1. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx =$

2. $\displaystyle\int_0^\infty x^2 e^{-x} dx =$

3. $\displaystyle\int \frac{dx}{1+\sqrt{x+1}} =$

4. $\displaystyle\int \frac{x-1-x^2}{x(1+x^2)}dx =$

5. The final answer here may still have an integral (which you don't need to evaluate)
   $\displaystyle\frac{d}{dx}\left(e^x \int_1^{\log x} \sqrt{\log t}\, dt\right) =$

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Problem 4. We'll say that a function f is "bigger" than a function g (and write $f \gg g$) if for every large enough x, $f(x) > g(x)$. Arrange the following functions by size:

$$f_1 = x^x, \qquad f_2 = 5\log x, \qquad f_3 = x^{\log x}, \qquad f_4 = x^5,$$

$$f_5 = \log(x^5), \qquad f_6 = (\log x)^x, \qquad f_7 = e^{\sqrt{x}}, \qquad f_8 = x^e.$$

Your answer should be a line like "$f_3 \gg f_8 \gg f_1 \gg \cdots \gg f_7 \gg f_2$", with a short justification for every comparison of the form "$f_3 \gg f_8$" or "$f_8 \gg f_1$" that you claim.

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Problem 5. Let f be a function which is differentiable n times at some point $a \in \mathbb{R}$.

1. Define "The nth Taylor polynomial $P_n = P_{n,a,f}(x)$ of f at a".

2. Prove that if P_n is the nth Taylor polynomial of f at a, then $\lim\limits_{x \to a} \dfrac{f(x) - P_n(x)}{(x-a)^n} = 0$.

3. For a certain function f it is known that $f(0) = 1$, $f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = f^{(5)}(0) = 0$ and that $f^{(6)}(0) = -2$. Prove that the point $x = 0$ is a local max of f.

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Problem 6. The "Cauchy Condensation Theorem" says that if a sequence (a_n) of positive numbers is decreasing then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

1. Prove the Cauchy Condensation Theorem.
2. Use the Cauchy Condensation Theorem to show that $\sum \frac{1}{n}$ diverges.
3. Use the Cauchy Condensation Theorem to show that $\sum \frac{1}{n \log_2 n}$ diverges.