Math 157 Analysis I — Solution of Term Exam 2

web version:
http://www.math.toronto.edu/~drorbn/classes/0405/157AnalysisI/TE2/Solution.html

The results. 82 students took the exam; the average grade was 45.3 and the standard deviation was about 25.

Required thought and response. These results are disappointing. What went wrong? You are **required** to think about it and send me your thoughts, by email, via CCNET or using the feedback form on the class' web site. Did something go wrong with the way you studied? In your opinion, was the exam unfair? Did I make serious mistakes in teaching the material? Did we sink into a routine and forgot to see the bigger picture? Anything else?

The goal of this exercise is to improve things. Be constructive! Don't just swallow or throw dirt, that won't help anyone. An indication that X went wrong is fine, but it's better if it comes along with "and Y could have fixed it".

As always, anonymous messages are fine (though signed messages are better). I guess this means that I cannot verify that you all do this exercise. Yet it remains morally required, for the benefit of everybody.

The due date for this task is next Friday, December 8, at 5PM. I may or may not prepare a synopsis of your responses (with all identifying details removed) for distribution as a handout early in the next semester.

Problem 1. Let f and g be continuous functions defined on all of $\mathbb{R}$.

1. Prove that if $f(a) \neq g(a)$ for some $a \in \mathbb{R}$, then there is a number $\delta > 0$ such that $f(x) \neq g(x)$ whenever $|x - a| < \delta$.

2. Prove that if two continuous functions are equal over the rationals then they are always equal. That is, if $f(r) = g(r)$ for every $r \in \mathbb{Q}$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Solution. (Graded by Derek Krepski)

1. Let $h(x) := f(x) - g(x)$. Then h is continuous and $h(a) \neq 0$. Set $\epsilon = |h(a)|/2$. Then $\epsilon > 0$, so using the continuity of h find $\delta > 0$ so that $|h(x) - h(a)| < \epsilon$ whenever $|x - a| < \delta$. Now if $|x - a| < \delta$ then $|h(x) - h(a)| < \epsilon = |h(a)|/2$, so $|h(x)| > |h(a)|/2$ and in particular $|h(x)| > 0$. but this means that when $|x - a| < \delta$ we have $h(x) \neq 0$ and so $f(x) \neq g(x)$.

2. Assume $f(r) = g(r)$ for every $r \in \mathbb{Q}$ and let a be an arbitrary real number. Assume $f(a) \neq g(a)$. Using part 1 of this question find $\delta > 0$ such that $f(x) \neq g(x)$ whenever $|x - a| < \delta$. Between any two different real numbers there is a rational number, so find $r \in \mathbb{Q}$ so that $a - \delta < r < a + \delta$. But then $|r - a| < \delta$ and so $f(r) \neq g(r)$, but $f(r) = g(r)$ as $r \in \mathbb{Q}$. This is a contradiction, so it can't be that $f(a) \neq g(a)$. Thus for every $a \in \mathbb{R}$ we have that $f(a) = g(a)$.

Problem 2. Let f be a continuous function defined on all of $\mathbb{R}$, and assume that $f(x)$ is rational for every $x \in \mathbb{R}$. Prove that f is a constant function.
Solution. (Graded by Derek Krepski) Assume by contradiction that $f(a) \neq f(b)$ for some real numbers $a < b$. Between any two real numbers there is an irrational number, so let s be some irrational number between $f(a)$ and $f(b)$. By the intermediate value theorem there is some $x \in (a, b)$ with $f(x) = s$. But then $f(x)$ is irrational, contradicting the assumption that $f(x)$ is rational for every $x \in \mathbb{R}$. Thus no such pair a, b exists and f must be a constant function.
Problem 3. We say that a function f is *locally bounded* on some interval I if for every $x \in I$ there is an $\epsilon > 0$ so that f is bounded on $I \cap (x - \epsilon, x + \epsilon)$. Let f be a locally bounded function on the interval $[a, b]$ and let $A = \{x \in [a, b] : f \text{ is bounded on } [a, x]\}$ and $c = \sup A$.

1. Justify the definition of c: How do we know that $\sup A$ exists?

2. Prove that $c > a$.

3. Prove that $c = b$.

4. Prove that $b \in A$.

5. Can you summarize these results with one catchy phrase?

Solution. (Graded by Shay Fuchs)

1. f is certainly bounded on the set $[a, a]$, so $a \in A$ and A is non-empty. Also, all elements of A are between a and b, hence A is bounded from above by b. So by P13 A has a least upper bound — in other words, $\sup A$ exists.

2. As f is locally bounded and as $a \in [a, b]$, there is some $\epsilon > 0$ for which f is bounded on $[a, a + \epsilon)$. But then f is bounded (with the same or smaller bound) on $[a, a + \frac{\epsilon}{2}]$, and so $a + \frac{\epsilon}{2} \in A$ and hence $c \geq a + \frac{\epsilon}{2} > a$.

3. Assume by contradiction that $c < b$. As f is locally bounded and as $c > a$, there is some $\epsilon > 0$ for which $(c-\epsilon, c+\epsilon) \subset [a,b]$ and f is bounded on $(c-\epsilon, c+\epsilon)$ (say by M_1). As $c = \sup A$ there is some $d \in A$ with $d > c - \epsilon$ and then by the definition of A, f is bounded by some constant M_2 on $[a,d]$. It follows that f is bounded by $\max(M_1, M_2)$ on $(c-\epsilon, c+\epsilon) \cup [a,d] \supset [a, c+\frac{\epsilon}{2}]$ and so $c + \frac{\epsilon}{2} \in A$, contradicting the fact that $c = \sup A$. So $c < b$ is false and hence $c \geq b$. But as b is an upper bound of A and c is a least upper bound of A, we also have that $c \leq b$ and hence $c = b$.

4. As f is locally bounded and as $b \in [a,b]$, there is some $\epsilon > 0$ for which f is bounded on $(b-\epsilon, b]$. Like before, f is also bounded on some $[a,d]$ with $d > b - \epsilon$, and hence on $[a,d] \cup (b-\epsilon, b] = [a,b]$. So by the definition of A we have that $b \in A$.

5. "On a closed interval, a locally bounded function is bounded".

Problem 4.

1. Define "f is differentiable at a".

2. Prove that if f is differentiable at a then it is also continuous at a.

Solution. (Graded by Brian Pigott)

1. f is said to be differentiable at a if f is defined in a neighborhood of a and $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists (and if this limit exists, it is called $f'(a)$).

2. If f is differentiable at a then (using the theorems about limits of sums and products),

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + (x-a)\frac{f(x)-f(a)}{x-a} \\
&= f(a) + \left(\lim_{x\to a} x - a\right)\left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right) \\
&= f(a) + 0 \cdot f'(a) = f(a).
\end{aligned}$$

   So by the definition of continuity, f is continuous at a.

Problem 5. Draw an approximate graph of the function $f(x) = \dfrac{x^2}{x^2-1}$ making sure to clearly indicate (along with clear justifications) the domain of definition of f, its x-intercepts and its y-intercepts (if any), the behaviour of f at

$\pm\infty$ and near points at which f is undefined (if any), intervals on which f is increasing/decreasing, its local minima/maxima (if any) and intervals on which f is convex/concave.

Solution. (Graded by Brian Pigott)

1. f is defined whenever the denominator x^2-1 is non-zero. That is, whenever $x \neq \pm 1$.

2. The x-intercepts are when $f(x)=0$. The only solution to that is when $x=0$, and that point is also the y-intercept.

3. $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$ and so $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} 1 + \frac{1}{x^2-1} = 1$. Also, as $x \to \pm 1$, the numerator of f goes to 1, so its behaviour is determined by the behaviour of its denominator, which is positive for $|x|>1$, negative for $|x|<1$ and zero for $|x|=1$. Hence $\lim_{x\to -1^-} = +\infty$, $\lim_{x\to -1^+} = -\infty$, $\lim_{x\to 1^-} = -\infty$ and $\lim_{x\to 1^+} = +\infty$.

4. $f'(x) = \frac{-2x}{(x^2-1)^2}$. The denominator here is always positive, so $f'(x)>0$ when $x<0$ and so f is increasing on $(-\infty,-1)$ and on $(-1,0)$ and $f'(x)<0$ when $x>0$ and so f is decreasing on $(0,1)$ and on $(1,\infty)$.

5. Comparing the intervals of decrease/increase, we find that f has a local max at $x=0$, and then $f(x)=0$.

6. $f''(x) = \frac{2+6x^2}{(1-x^2)^3}$. Here the numerator is always positive so the sign is determined by the denominator. Hence $f''>0$ and f is convex on $(-\infty,-1)$ and on $(1,\infty)$, while $f''<0$ and f is concave on $(-1,1)$.

In summary, the graph of f is roughly as follows: