Math 157 Analysis I — Solution of Term Exam 3

web version:
http://www.math.toronto.edu/~drorbn/classes/0405/157AnalysisI/TE3/Solution.html

Problem 1.

1. Compute $\int_0^1 \sqrt{x}\,dx$.
2. Compute $\int_0^\pi \sin x\,dx$.
3. For $x \geq 0$, compute $\frac{d}{dx}\int_{x^3}^{157} \sqrt{t}\,dt$.

Solution. (Graded by Shay Fuchs)

1. To use the second fundamental theorem of calculus we are looking for a function f for which $f' = \sqrt{x} = x^{1/2}$. The most obvious guess is $f(x) = x^{3/2}$, but this is off by a factor of 3/2, for $(x^{3/2})' = \frac{3}{2} = x^{1/2}$. So a good answer would be $f(x) = \frac{2}{3}x^{3/2}$. Now
$$\int_0^1 \sqrt{x}\,dx = \int_0^1 f'(x)dx = f|_0^1 = f(1) - f(0) = \frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2} = \frac{2}{3}.$$
2. Likewise choose $f(x) = -\cos x$ to get $f'(x) = \sin x$, and so using the second fundamental theorem of calculus,
$$\int_0^\pi \sin x\,dx = \int_0^\pi f'(x)dx = f(\pi) - f(0) = -\cos\pi - (-\cos 0) = 2.$$
3. Let $g(y) = \int_{157}^y \sqrt{t}\,dt$ and let $f(x) = x^3$. Using the first fundamental theorem of calculus, $g'(y) = \sqrt{y}$. So using the chain rule,
$$\begin{aligned} \frac{d}{dx}\int_{x^3}^{157}\sqrt{t}\,dt &= \frac{d}{dx}\left(-\int_{157}^{x^3}\sqrt{t}\,dt\right) = -(g\circ f)' \\ &= -g'(f(x))f'(x) = -\sqrt{x^3}3x^2 = -3x^{7/2}. \end{aligned}$$

Problem 2.

1. Perhaps using L'Hôpital's law, compute $\lim_{x\to 0}\dfrac{\sin x}{x}$ and $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$.

2. Use these results to give educated guesses for the values of $\sin 0.1$ and $\cos 0.1$ (no calculators, please).

Solution. (Graded by Shay Fuchs)

1. $\sin x$ is differentiable at 0 and $\sin 0 = 0$. So

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin x - \sin 0}{x} = \sin' 0 = \cos 0 = 1.$$

   (L'Hôpital's law also works and gives the same result).

   The second limit is of the form $\frac{0}{0}$ so we can use L'Hôpital:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)'}{(x^2)'} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{2}.$$

2. 0.1 is close to 0, so $\frac{\sin 0.1}{0.1} \sim \lim_{x\to 0}\frac{\sin x}{x} = 1$. Multiplying both sides by 0.1 we get $\sin 0.1 \sim 0.1$.

   Likewise, $\frac{1-\cos 0.1}{0.1^2} \sim \frac{1}{2}$, so $1-\cos 0.1 \sim \frac{1}{2}0.1^2 = 0.005$, so $\cos 0.1 \sim 0.995$.

Problem 3.

1. State the "one partition for every ϵ" criterion of the integrability of a bounded function f defined on an interval $[a, b]$.

2. Let f be an increasing function on $[0, 1]$ and let P_n be the partition defined by $t_i = i/n$, for $i = 0, 1, \ldots, n$. Write simple formulas for $U(f, P_n)$ and for $L(f, P_n)$.

3. Under the same conditions, write a very simple formula for $U(f, P_n) - L(f, P_n)$.

4. Prove that an increasing function on $[0, 1]$ is integrable.

Solution. (Graded by Derek Krepski)

1. A bounded function f defined on an interval $[a, b]$ is integrable iff for every $\epsilon > 0$ there is a partition P of $[a, b]$ for which $U(f, P) - L(f, P) < \epsilon$.

2. As f is increasing, $m_i^{P_n} = \inf_{[t_{i-1},t_i]} f(x) = f(t_{i-1})$ and $M_i^{P_n} = \sup_{[t_{i-1},t_i]} f(x) = f(t_i)$. Thus

$$U(f, P_n) = \sum_{i=1}^{n} M_i^{P_n}(t_i - t_{i-1}) = \sum_{i=1}^{n} f(t_i)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n}).$$

   Likewise, $L(f, P_n) = \frac{1}{n}\sum_{i=1}^{n} f(t_{i-1}) = \frac{1}{n}\sum_{i=1}^{n} f(\frac{i-1}{n})$.

3.

$$U(f,P_n) - L(f,P_n) = \frac{1}{n}\sum_{i=1}^{n} f(t_i) - \frac{1}{n}\sum_{i=1}^{n} f(t_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}(f(t_i) - f(t_{i-1}))$$

using telescopic summation this is

$$= \frac{1}{n}(f(t_n) - f(t_0)) = \frac{1}{n}(f(1) - f(0)).$$

4. Since f is increasing, f is bounded (with upper bound $f(1)$ and lower bound $f(0)$). So using the criterion of part 1, to show that f is integrable it is enough to show that for every $\epsilon > 0$ there is a partition P of $[0,1]$ for which $U(f,P) - L(f,P) < \epsilon$. Indeed, let $\epsilon > 0$ be given. Choose n so big so that $\frac{1}{n}(f(1)-f(0)) < \epsilon$, and then the partition $P = P_n$ of before satisfies $U(f,P_n) - L(f,P_n) = \frac{1}{n}(f(1) - f(0)) < \epsilon$, as required.

Problem 4.

1. Show that the function $f(x) = 3x - x^3$ is monotone on the interval $[-1,1]$.
2. Deduce that for every $c \in [-2,2]$ the equation $3x - x^3 = c$ has a unique solution x in the range $-1 \le x \le 1$.
3. For $c \in [-2,2]$, let $g(c)$ be the unique x in the range $-1 \le x \le 1$ for which $3x - x^3 = c$. Write a formula for $g'(c)$ and simplify it as much as you can. Your end result may still contain $g(c)$ in it, but not f, f' or g'.

Solution. (Graded by Brian Pigott)

1. $f'(x) = 3 - 3x^2 = 3(1-x^2)$. On $(-1,1)$ we know that $x^2 < 1$, so $f'(x) > 0$. So f is increasing on $[-1,1]$.
2. By the theorem about the existence of inverses of monotone functions, f has an inverse on $[-1,1]$ and it is defined on $[f(-1), f(1)] = [-2,2]$. This precisely means that for $c \in [-2,2]$ the equation $3x - x^3 = c$ (which defines $f^{-1}(c)$) has a unique solution with x in the range $-1 \le x \le 1$.
3. By the theorem about the derivative of an inverse function,

$$g'(c) = \frac{1}{f'(g(c))} = \frac{1}{3(1-g(c)^2)}.$$

The results. 67 students took the exam; the average grade was 77.7 and the standard deviation was about 22.