

## Covering Spaces in One Swoosh

web version: <http://www.math.toronto.edu/~drorbn/classes/0405/Topology/CoveringSpaces/CoveringSpaces.html>

Let  $B$  be a topological space and let  $\mathcal{C}(B)$  be the category of covering spaces of  $B$ : The category whose objects are coverings  $X \rightarrow B$  and whose morphisms are maps between such coverings that commute with the covering projections — a morphism between  $p_X : X \rightarrow B$  and  $p_Y : Y \rightarrow B$  is a map  $\alpha : X \rightarrow Y$  so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ p_X \searrow & & \swarrow p_Y \\ & B & \end{array}$$

is commutative.

Every topologists' highest hope is to find that her/his favourite category of topological objects is equivalent to some category of easily understood algebraic objects. The following theorem realizes this dream in full in the case of the category  $\mathcal{C}(B)$  of covering spaces of any reasonable base space  $B$ :

**Theorem 1 (Classification of covering spaces)** • *If  $B$  is connected and locally connected with base point  $b$  and fundamental group  $G = \pi_1(B, b)$ , then the map which assigns to every covering  $p : X \rightarrow B$  its fiber  $p^{-1}(b)$  over the basepoint  $b$  induces a functor  $\mathcal{F}$  from the category  $\mathcal{C}(B)$  of coverings of  $B$  to the category  $\mathcal{S}(G)$  of  $G$ -sets — sets with a right  $G$ -action and set maps that respect the  $G$  action.*

- *If in addition  $B$  is semi-locally simply connected then the functor  $\mathcal{F}$  is an equivalence of categories. (In fact, this is iff).*

If indeed the categories  $\mathcal{C}(B)$  and  $\mathcal{S}(G)$  are equivalent, one should be able to extract everything topological about a covering  $p : X \rightarrow B$  from its associated  $G$ -set  $\mathcal{F}(X) = p^{-1}(b)$ . The following theorem shows this to be right in at least two ways:

**Theorem 2** • *The set of connected components of  $X$  is in a bijective correspondence with the set of orbits of  $G$  in  $\mathcal{F}(X)$ .*

- *Let  $x \in \mathcal{F}(X) = p^{-1}(b)$  be a basepoint for  $X$  that covers the basepoint  $b$  of  $B$ . Then the fundamental group  $\pi_1(X, x)$  is isomorphic via the projection  $p_*$  into  $G = \pi_1(B, b)$  to the stabilizer group  $\{h \in G : xh = x\}$  of  $x$  in  $\mathcal{F}(X)$ .*

(Both assertions of this theorem can be sharpened to deal with morphisms as well, but we will not bother to do so).

Ok. Every math technician can spend some time and effort and understand the statements and (only then) the proofs of these two theorems. Your true challenge is to digest the following statement:

**All there is to know about covering spaces follows from these two theorems**

In particular, the following facts are all simple algebraic corollaries of these theorems:

**Corollary 3** *If  $X$  is connected then its covering number (= “number of decks”) is equal to the index of  $H = p_*\pi_1(X)$  in  $G = \pi_1(B)$ , and the decks of  $X$  are in a non-canonical correspondence with the left cosets  $H \backslash G$  of  $H$  in  $G$ .*

**Corollary 4** *If  $B$  is semi-locally simply connected, there exists a unique (up to base-point-preserving isomorphism) “universal covering space  $U$  of  $B$ ” (a connected and simply connected covering  $U$ ).*

**Corollary 5** *The group of automorphisms of the universal covering  $U$  is equal to  $G = \pi_1(B)$ .*

**Corollary 6**  $\pi_1(S^1) = \mathbb{Z}$ .

**Corollary 7**  $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ .

**Corollary 8** *If  $B$  is semi-locally simply connected, then for every  $H < G = \pi_1(B)$  there is a unique (up to base-point-preserving isomorphism) connected covering space  $X$  with  $p_*\pi_1(X) = H$ .*

**Corollary 9** *If  $X_i$  for  $i = 1, 2$  are connected coverings of  $B$  with groups  $H_i = p_{i*}\pi_1(X_i)$  and if  $H_1 < H_2$  then  $X_1$  is a covering of  $X_2$  of covering number  $(H_2 : H_1)$ .*

**Corollary 10** *If  $B$  is semi-locally simply connected there is a bijection between conjugacy classes of subgroups of  $G = \pi_1(B)$  and unbased connected coverings of  $B$ .*

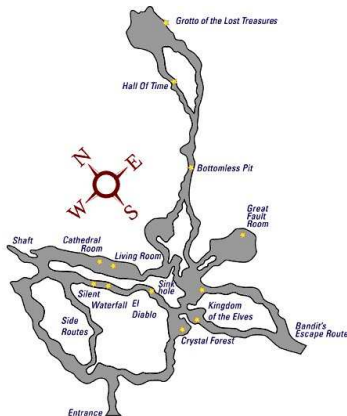
**Corollary 11** *A connected covering  $X$  is normal (for any  $x_1, x_2 \in p^{-1}(b)$  there is an automorphism  $\tau$  of  $X$  with  $\tau x_1 = x_2$ ) iff its group  $p_*\pi_1(X)$  is normal in  $G = \pi_1(B)$ .*

**Corollary 12** *If  $X$  is a connected covering of  $B$  and  $H = p_*\pi_1(X)$ , then  $\text{Aut}(X) = N_G(H)/H$  where  $N_G(H)$  is the normalizer of  $H$  in  $G$ .*

**Proposition 13** *If I forgot anything, it follows too.*

### Steps in the proofs of Theorem 1 and 2.

1. Use path liftings to construct a right action of  $G$  on  $p^{-1}(b)$ .
2. Show that this is indeed a group action and that morphisms of coverings induce morphisms of right  $G$ -sets.



A map of Colossal Cave, New Mexico, from <http://www.colossalcave.com/cavetour.html>.

3. Start the construction of an “inverse” functor  $\mathcal{G}$  of  $\mathcal{F}$ : Use spelunking (cave exploration) to construct a universal covering  $U$  of  $B$ , if  $B$  is semi-locally simply connected.
4. Show that  $\mathcal{F}(U) = G$ .
5. Use the construction of  $U$  or the general lifting property for covering spaces to show that there is a left action of  $G$  on  $U$ .
6. For a general right  $G$ -set  $S$  set  $\mathcal{G}(S) = S \times_G U = \{(s, u) \in S \times U\} / (sg, u) \sim (s, gu)$  and show that  $\mathcal{G}(S)$  is a covering of  $B$  and  $\mathcal{F}(\mathcal{G}(S)) = S$ .
7. Show that  $\mathcal{G}$  is compatible with maps between right  $G$ -sets.
8. Understand the relationship between connected components and orbits.
9. Prove Theorem 2.
10. Use the existence and uniqueness of lifts to show that  $\mathcal{G} \circ \mathcal{F}$  is equivalent to the identity functor (working connected component by connected component).

**A Deep Thought Question.** We’ll get there when it’s time, but meanwhile, think on your own: What does it at all mean “ $\mathcal{G} \circ \mathcal{F}$  is equivalent to the identity functor” (and first, why can’t it simply **be** the identity functor)? And even harder, what does it at all mean for two categories to be “equivalent”? If you answer this question correctly, you’ll probably re-invent the notions of “natural transformation between two functors” and “natural equivalence”, that gave the historical impetus for the development of category theory.

#### Category theory

From Wikipedia, the free encyclopedia ([http://en.wikipedia.org/wiki/Category\\_theory](http://en.wikipedia.org/wiki/Category_theory)).

#### Background

A category attempts to capture the essence of a class of related mathematical objects, for instance the class of groups. Instead of focusing on the individual objects (groups) as has been done traditionally, the morphisms — i.e. the structure-preserving maps between these objects — are emphasized. In the example of groups, these are the group homomorphisms. Then it becomes possible to relate different categories by functors, generalizations of functions which associate to every object of one category an object of another category and to every morphism in the first category a morphism in the second. Very commonly, certain “natural constructions”, such as the fundamental group of a topological space, can be expressed as functors. Furthermore, different such constructions are often “naturally related” which leads to the concept of natural transformation, a way to “map” one functor to another. Throughout mathematics, one encounters “natural isomorphisms”, things that are (essentially) the same in a “canonical way”. This is made precise by special natural transformations, the natural isomorphisms.

#### Historical notes

Categories, functors and natural transformations were introduced by Samuel Eilenberg and Saunders Mac Lane in 1945. Initially, the notions were applied in topology, especially algebraic topology, as an important part of the transition from homology (an intuitive and geometric concept) to homology theory, an axiomatic approach. It has been claimed, for example by or on behalf of Ulam, that comparable ideas were current in the later 1930s in the Polish school.

Eilenberg/Mac Lane have said that their goal was to understand natural transformations; in order to do that, functors had to be defined; and to define functors one needed categories.

The subsequent development of the theory was powered first by the computational needs of homological algebra; and then by the axiomatic needs of algebraic geometry, the field most resistant to the Russell-Whitehead view of united foundations. General category theory — an updated universal algebra with many new features allowing for semantic flexibility and higher-order logic — came later; it is now applied throughout mathematics.

Special categories called topoi can even serve as an alternative to axiomatic set theory as the foundation of mathematics. These broadly-based foundational applications of category theory are contentious; but they have been worked out in quite some detail, as a commentary on or basis for constructive mathematics. One can say, in particular, that axiomatic set theory still hasn’t been replaced by the category-theoretic commentary on it, in the everyday usage of mathematicians. The idea of bringing category theory into earlier, undergraduate teaching (signified by the difference between the Birkhoff-Mac Lane and later Mac Lane-Birkhoff abstract algebra texts) has hit noticeable opposition.

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with application to the theory of functional programming and domain theory, all in a setting of a cartesian closed category as non-syntactic description of a lambda calculus. At the very least, the use of category theory language allows one to clarify what exactly these related areas have in common (in an abstract sense).