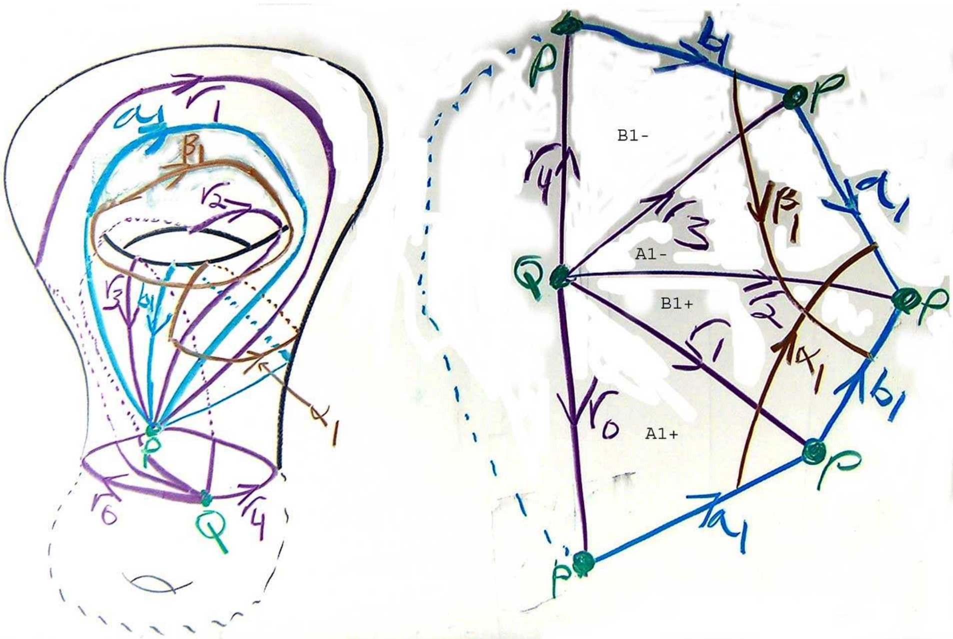
A Cup Product Example

web version: http://www.math.toronto.edu/~drorbn/classes/0405/Topology/CupExample/CupExample.html



From this picture (drawn with help from Jacob Tsimerman) we can read the following:

$$\alpha_1^\dagger = a_1^\star + r_1^\star + r_2^\star, \qquad -\beta_1^\dagger = b_1^\star + r_3^\star + r_2^\star.$$
$$\partial_2 A_{1+} = r_0, \quad \partial_0 A_{1+} = a_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger (A_{1+}) = \alpha_1(\partial_2 A_{1+})\beta_1(\partial_0 A_{1+}) = 0 \cdot 0 = 0.$$
$$\partial_2 B_{1+} = r_1, \quad \partial_0 B_{1+} = b_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger (B_{1+}) = \alpha_1(\partial_2 B_{1+})\beta_1(\partial_0 B_{1+}) = 1 \cdot (-1) = -1.$$
$$\partial_2 A_{1-} = r_3, \quad \partial_0 A_{1-} = a_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger (A_{1-}) = \alpha_1(\partial_2 A_{1-})\beta_1(\partial_0 A_{1-}) = 0 \cdot 0 = 0.$$
$$\partial_2 B_{1-} = r_4, \quad \partial_0 B_{1-} = b_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger (B_{1-}) = \alpha_1(\partial_2 B_{1-})\beta_1(\partial_0 B_{1-}) = 0 \cdot (-1) = 0.$$

So $\alpha_1^\dagger \cup \beta_1^\dagger = -B_{1+}^\star$ is a generator of H^2.

Exercise. Verify that $\beta_1^\dagger \cup \alpha_1^\dagger = -A_{1-}^\star$ is also a generator of H^2, but note that in H^2 we have $B_{1+}^\star = -A_{1-}^\star$ so the cup product is not commutative!

The Hopf Fibration as drawn by Penrose and Rindler: