Final Exam

University of Toronto, April 29, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve 2 of the 3 problems in part A and 4 of the 6 problems in part B. Each problem is worth 17 points, to a maximal total grade of 102. If you solve more than the required 2 in 3 and 4 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve 5 of the 6 problems in part B, do not solve anything in part A. Each problem is worth 20 points. If you solve more than the required 5 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Good Luck!

Part A

Problem 1. Let X be a topological space.

1. Define the phrase "X is Hausdorff".
2. Define the phrase "X is normal".
3. Define the phrase "X is compact".
4. Prove that if X is compact and Hausdorff, it is normal.

Problem 2. Let X be a metric space.

1. Define the phrase "X is complete".
2. Define the phrase "X is totally bounded".
3. Prove that if X is totally bounded and complete than every sequence in X has a convergent subsequence.

Problem 3.

1. State the Van Kampen theorem in full.
2. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk in the complex plane and let Y be its quotient by the relation $z \sim ze^{2\pi i/3}$, for $|z| = 1$. Compute $\pi_1(Y)$.

Part B

Problem 4.

1. Let $p : X \to B$ be covering map and let $f : Y \to B$ be a continuous map. State in full the lifting theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a lift of f to a map $\tilde{f} : Y \to X$ such that $f = p \circ \tilde{f}$.
2. Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{it}$. Is it true that every map $f : \mathbb{RP}^2 \to S^1$ can be lifted to a map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}$ such that $f = p \circ \tilde{f}$? Justify your answer.

Problem 5. Let M be an n-dimensional topological manifold (a space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$), and let p be a point in M.

1. Show that p has a neighborhood U for which $H_k(M-p, U-p)$ is isomorphic to $\tilde{H}_k(M)$ for all k, and so that U is homeomorphic to a ball.
2. Write the long exact sequence corresponding to the pair $(M-p, U-p)$.
3. Prove that $\tilde{H}_k(M-p)$ is isomorphic to $\tilde{H}_k(M)$ for $k < n-1$.

Problem 6.

1. Present the space $X = S^2 \times S^4$ as a CW complex.

2. Calculate the homology of X. (I.e., calculate $H_k(X)$ for all k).

3. What is the minimal number of cells required to present X as a CW complex? Justify your answer.

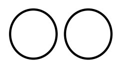
Problem 7.

1. Define the *degree* $\deg \Phi$ of a continuous map $\Phi : T^2 \to S^2$.

2. Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be two continuous maps such that $\gamma_1(S^1) \cap \gamma_2(S^1) = \emptyset$. Let $\Phi_{\gamma_1,\gamma_2} : T^2 = S^1 \times S^1 \to S^2$ be defined by

$$\Phi_{\gamma_1,\gamma_2}(z_1, z_2) := \frac{\gamma_2(z_2) - \gamma_1(z_1)}{|\gamma_2(z_2) - \gamma_1(z_1)|},$$

for $z_1, z_2 \in S^1$. Prove that the degree $l(\gamma_1, \gamma_2) := \deg \Phi_{\gamma_1,\gamma_2}$ is invariant under homotopies of γ_1 and γ_2 throughout which γ_1 and γ_2 remain disjoint. (I.e., homotopies $\gamma_{1,t}$ and $\gamma_{2,t}$ for which $\gamma_{1,t}(S^1) \cap \gamma_{2,t}(S^1) = \emptyset$ for all t).

3. Compute (without worrying about signs, but otherwise with justification) the degree $l(\gamma_1, \gamma_2)$ where γ_1 and γ_2 are given by the picture .

4. Compute (without worrying about signs, but otherwise with justification) the degree $l(\gamma_1, \gamma_2)$ where γ_1 and γ_2 are given by the picture .

Problem 8.

1. State the theorem about the homology of the complement of an embedded disk in $\mathbb{R}^n$.

2. State the theorem about the homology of the complement of an embedded sphere in $\mathbb{R}^n$.

3. Prove that the first of these two theorems implies the second.

Problem 9. A chain complex A is said to be "acyclic" if its homology vanishes (i.e., if it is an exact sequence). Let C be a subcomplex of some chain complex B.

1. Show that if C is acyclic then the homology of B is isomorphic to the homology of B/C (so C "doesn't matter").

2. Show that if B/C is acyclic then the homology of B is isomorphic to the homology of C (so "the part of B out of C" doesn't matter).

3. If B is acyclic, can you say anything about the relation between the homology of C and the homology of B/C?

Good Luck!