Dreams on the (Co)Homology of Manifolds

web version:
http://www.math.toronto.edu/~drorbn/classes/0405/Topology/ManifoldDreams/ManifoldDreams.html

The Context. Let M be an n-dimensional manifold (a topological space that locally looks like $\mathbb{R}^n$) and let R be a ring.

Dream 1 *The following should be a list of spaces and their duals; there ought to be no (co)homology beyond that list:*

$$H_0(M) \qquad H_1(M) \qquad H_2(M) \qquad \cdots \qquad H_{n-1}(M) \qquad H_n(M)$$

$$H^0(M) \qquad H^1(M) \qquad H^2(M) \qquad \cdots \qquad H^{n-1}(M) \qquad H^n(M)$$

Dream 2 *There should be an "intersection pairing"*

$$H_k \times H_l \to H_{k+l-n},$$

induced from the intersection pairing of submanifolds which ought to satisfy $\partial(\sigma \cap \lambda) = (\partial\sigma) \cap \lambda + \sigma \cap (\partial\lambda)$.

Dream 3 *In particular, there should be a pairing*

$$H_k \times H_{n-k} \to H_0 = R,$$

so with some further optimism, H_k *ought to be the same as* $(H_{n-k})^\star = H^{n-k}$. *(And why not call that "Poincaré Duality"?)*

Dream 4 H_n *should be* R *(and hence* H^n *should be* R *as well).*

Dream 5 *There should be a "cap product"*

$$\cap : H_k \times H^l \to H_{k-l}.$$

Dream 6 *There should be a "cup product"*

$$\cup : H^k \times H^l \to H^{k+l},$$

and so $H^\star := \oplus_k H^k$ *ought to be a ring!*

Jules Henri Poincaré
(from http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Poincare.html)

Warning. Dreams are based on reality. Often, distorted reality.