A Sample Final Exam

University of Toronto, April 12, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve 2 of the 3 problems in part A and 4 of the 6 problems in part B. Each problem is worth 17 points, to a maximal total grade of 102. If you solve more than the required 2 in 3 and 4 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve 5 of the 6 problems in part B, do not solve anything in part A. Each problem is worth 20 points. If you solve more than the required 5 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Good Luck!

Part A

Problem 1. Let X be a topological space.

1. Define the "product topology" on $X \times X$.
2. Prove that if X is compact then so is $X \times X$.
3. Prove that the "folding of X along the diagonal", $S^2 X := X \times X / (x, y) \sim (y, x)$ is also compact.

Problem 2. Let X be a compact metric space and let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X. Show that there exists $\epsilon > 0$ such that for every $x \in X$ there exists $\alpha \in A$ such that the ϵ-ball centred at x is contained in U_α. (ϵ is called a *Lebesgue number* for the covering.)

Problem 3.

1. Compute $\pi_1(\mathbb{RP}^2)$.
2. A topological space X_f is obtained from a topological space X by gluing to X an n-dimensional cell e^n using a continuous gluing map $f : \partial e^n = S^{n-1} \to X$, where $n \geq 3$. Prove that obvious map $\iota : \pi_1(X) \to \pi_1(X_f)$ is an isomorphism.
3. Compute $\pi_1(\mathbb{RP}^n)$ for all n.

Part B

Problem 4. Let $p : X \to B$ be a covering of a connected locally connected and semi-locally simply connected base B with basepoint b. Prove that if $p_\star \pi_1(X)$ is normal in $\pi_1(B)$ then the group of automorphisms of X acts transitively on $p^{-1}(b)$.

Problem 5. A topological space X_f is obtained from a topological space X by gluing to X an n-dimensional cell e^n using a continuous gluing map $f : \partial e^n = S^{n-1} \to X$, where $n \geq 2$. Show that

1. $H_m(X) \cong H_m(X_f)$ for $m \neq n, n-1$.
2. There is an exact sequence

$$0 \to H_n(X) \to H_n(X_f) \to H_{n-1}(S^{n-1}) \to H_{n-1}(X) \to H_{n-1}(X_f) \to 0.$$

Problem 6. Let T denote the (standard) 2-dimensional torus.

1. State the homology and cohomology of T including the ring structure. (Just state the results; no justification is required.)
2. Show that every map f from the sphere S^2 to T induces the zero map on cohomology. (Hint: cohomology flows against the direction of f).

Problem 7. For $n \geq 1$, what is the degree of the antipodal map on S^n? Give an example of a continuous map $f : S^n \to S^n$ of degree 2 (your exmple should work for every n). Explain your answers.

Problem 8.

1. State the "Salad Bowl Theorem".

2. State the "Borsuk-Ulam Theorem".

3. Prove that the latter implies the former.

Problem 9. Suppose

$$\begin{array}{ccccccccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & D' & \xrightarrow{d'} & E'
\end{array}$$

is a commutative diagram of Abelian groups in which the rows are exact and α, β, δ and ϵ are isomorphisms. Prove that γ is also an isomorphism.

Good Luck!

Warning: The real exam will be similar to this sample, to my taste. Your taste may be significantly different.