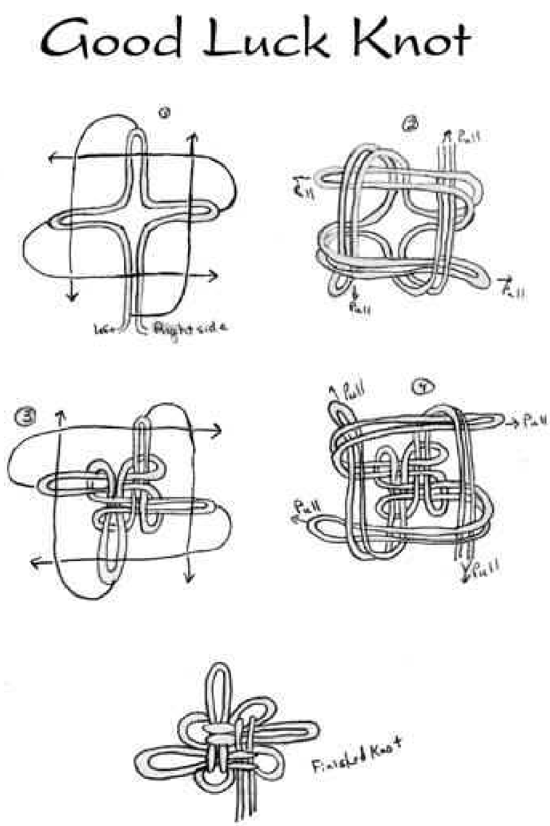
Math 1300Y Topology — Term Exam 1

University of Toronto, November 16, 2004

Solve 5 of the following 6 problems. Each problem is worth 20 points. If you solve more than 5 problems indicate very clearly which ones you want graded; otherwise a random one will be left out at grading and it may be your best one! You have an hour and 50 minutes. No outside material other than stationary is allowed.



(from http://www.mresource.com/Fiber/COEPart2/goodluckknot.htm)

Problem 1. Let X be an arbitrary topological space. Show that the diagonal $\Delta = \{(x,x) : x \in X\}$, taken with the topology induced from $X \times X$, is homeomorphic to X. (18 points for any correct solution. 20 points for a correct solution that does not mention the words "inverse image", "open set", "closed set" and/or "neighborhood".)

Problem 2. Let (X,d) be a connected metric space and let x and y be two different points of X.

1. Prove that if $0 \le r \le d(x,y)$ then the sphere of radius r around x, $S_r(x) := \{z : d(x,z) = r\}$, is non-empty.
2. Prove that the cardinality of X is at least as big as the continuum: $|X| \ge 2^{\aleph_0}$.

Problem 3.

1. Define "X is completely regular".
2. Prove that a topological space X can be embedded in a cube (a space of the form I^A, for some A) iff it is completely regular.

Problem 4.

1. Define "X is $T4$ (normal)".
2. For the purpose of this problem, we say that a topological space is $T4\frac{1}{4}$ if whenever A and B are disjoint closed subsets of X, there exist open sets U and V in X so that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Prove that if X is $T4$ then it is also $T4\frac{1}{4}$.

Problem 5. The "diameter" of a metric space (X,d) is defined to be $D_X := \sup\{d(x,y) : x, y \in X\}$.

1. Prove that a compact metric space has a finite diameter.
2. Prove that if X is a compact metric space, then there's a pair of points $x_0, y_0 \in X$ so that $D_X = d(x_0, y_0)$.

Problem 6. If f_n is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$, show that f is continuous at uncountably many points of $\mathbb{R}$.

Good Luck!