Term Exam 2

University of Toronto, March 8, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve one of the two problems in part A and three of the four problems in part B. Each problem is worth 25 points. If you solve more than the required 1 in 2 and 3 in 4, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have an hour and 50 minutes. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve the four problems in part B, do not solve anything in part A. Each problem is worth 25 points. You have an hour and 50 minutes. No outside material other than stationary is allowed.

Apology: due to my travel plans grading may be slow.

Good Luck!

Part A

Problem 1. Let X be a group with product $\star$.

1. What does it mean to say that "X is a topological group"?
2. If $\gamma_1 : I \to X$ and $\gamma_2 : I \to X$ are paths in X, define $\gamma_1 \star \gamma_2 : I \to X$ by $(\gamma_1 \star \gamma_2)(t) = \gamma_1(t) \star \gamma_2(t)$. Show that $[\gamma_1 \star \gamma_2] = [\gamma_1][\gamma_2]$ in $\pi_1(X)$.
3. Show that $\pi_1(X)$ is Abelian.

Problem 2.

1. State Van-Kampen's theorem.
2. Let X be the result of identifying every edge of a hexagon with its opposite in a parallel manner (to a total of 3 edge pair identifications). Compute $\pi_1(X)$. (The hexagon comes along with its interior, but the identification occurs only on the boundary).
3. (5 points bonus) Explain in a very convincing manner how X is homeomorphic to a well known space seen in class several times.

Part B

Problem 3. Let B be a connected, locally connected and semi-locally simply connected topological space with basepoint b.

1. State the classification theorem for the category of covering spaces of B.
2. Abstractly define "the universal covering U of B" using the classification theorem.
3. Use the classification theorem to show that any connected covering X of B is covered by U.

Problem 4.

1. Define "a homotopy between two morphisms f and g of chain complexes".
2. Show that homotopy of morphisms is an equivalence relation on the set of all morphisms between two given complexes.
3. Show that if $f : A \to B$ and $g : A \to B$ are homotopic morphisms of chain complexes A and B, and if $j : B \to C$ is another morphism of chain complexes, then $j \circ f$ is homotopic to $j \circ g$.

Problem 5. Let X and Y be disjoint topological spaces with basepoints x and y, respectively. Assume also that x has a neighborhood U that deformation retracts (i.e., contracts) to x and likewise that y has a neighborhood V that contracts to y. Recall that the wedge sum $X \vee Y$ is $X \cup Y / x \sim y$. What is the relationship between the homologies (reduced or not, your choice) of X, Y and $X \vee Y$? Prove your assertions. (Hint: it is a good idea to excise the "linkage point" $x \sim y$).

Problem 6. A 3-dimensional Δ-complex X is defined by

$$S_3 = \{t\} \xrightarrow{\partial_{0,1,2,3}} S_2 = \{f_0, f_1\} \xrightarrow{\partial_{0,1,2}} S_1 = \{e_0, e_1, e_2\} \xrightarrow{\partial_{0,1}} S_0 = \{v_0, v_1\},$$

with boundary maps $\partial_{0,1,2,3} t = (f_0, f_0, f_1, f_1)$, $\partial_{0,1,2} f_0 = (e_0, e_1, e_1)$, $\partial_{0,1,2} f_1 = (e_1, e_1, e_2)$, $\partial_{0,1} e_0 = (v_0, v_0)$, $\partial_{0,1} e_1 = (v_0, v_1)$ and $\partial_{0,1} e_2 = (v_1, v_1)$.

1. Write down the chain complex $C(X)$ (including the boundary maps).
2. Compute the homology groups $H_n(X)$ of X for $0 \le n \le 3$.
3. Can you identify $|X|$?

Good Luck!