A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial

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The Kauffman Bracket: $\langle \emptyset \rangle = 1$; $\quad \langle \bigcirc L \rangle = (q+q^{-1})\langle L \rangle$; $\quad \langle \times \rangle = \langle \underset{0-\text{smoothing}}{\asymp} \rangle - q \langle \underset{1-\text{smoothing}}{)(} \rangle$.

The Jones Polynomial: $\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$, where (n_+, n_-) count $(\times, \times)$ crossings.

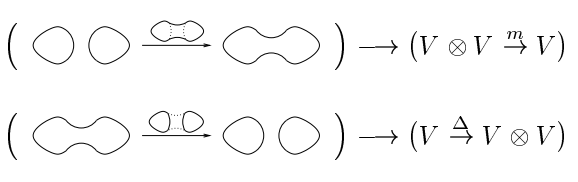
Khovanov's construction: $[\![L]\!]$ — a chain complex of graded $\mathbb{Z}$-modules;

$$[\![\emptyset]\!] = 0 \to \underset{\text{height } 0}{\mathbb{Z}} \to 0; \qquad [\![\bigcirc L]\!] = V \otimes [\![L]\!]; \qquad [\![\times]\!] = \text{Flatten}\left(0 \to \underset{\text{height } 0}{[\![\asymp]\!]} \to \underset{\text{height } 1}{[\![)(]\!]\{1\}} \to 0 \right);$$

$$\mathcal{H}(L) = \mathcal{H}\left(\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}\right)$$

$V = \text{span}\langle v_+, v_- \rangle$; $\qquad \deg v_\pm = \pm 1$; $\qquad q\dim V = q + q^{-1}$ $\quad$ with $\quad q\dim \mathcal{O} := \sum_m q^m \dim \mathcal{O}_m$;

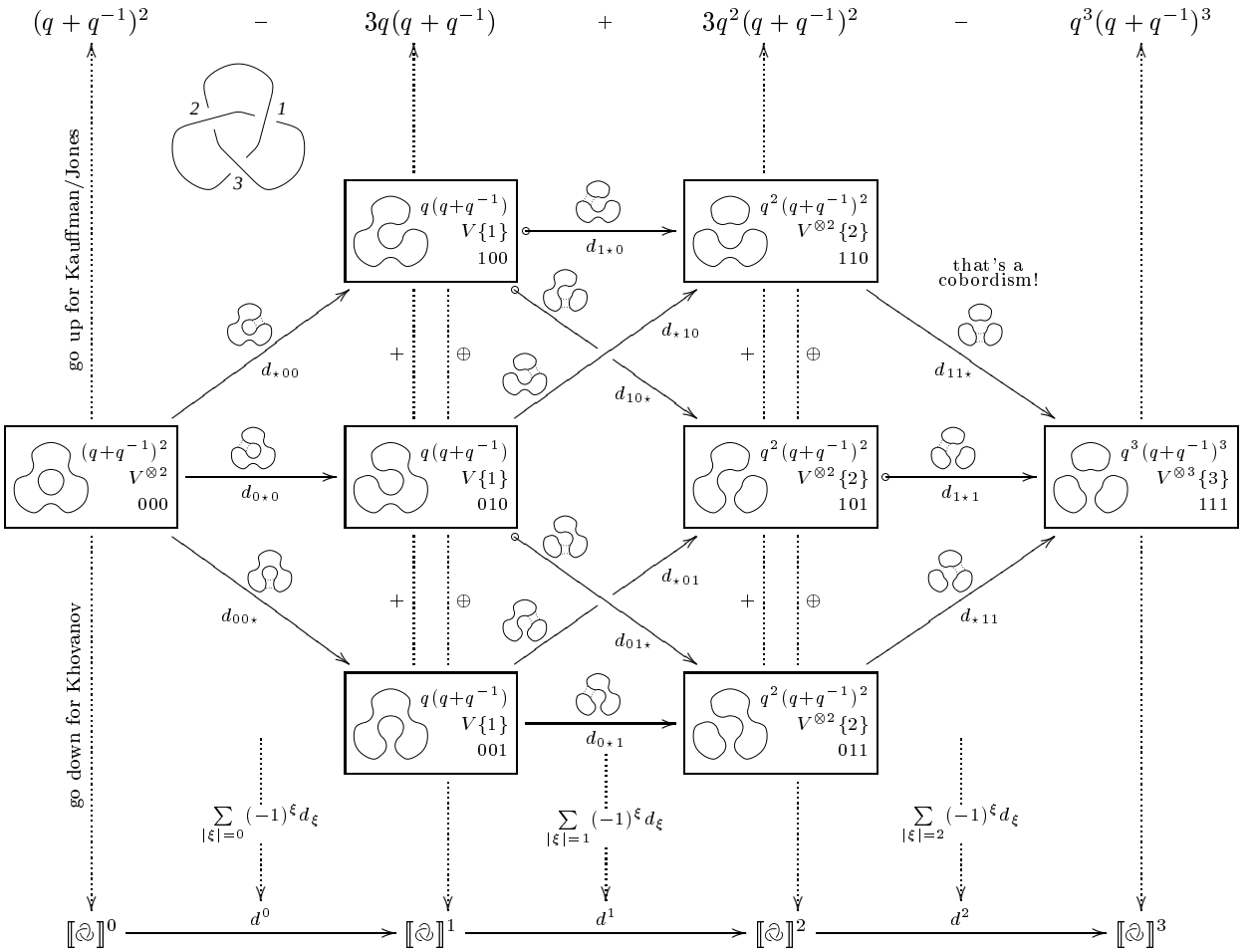
$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l} \quad \text{so} \quad q\dim \mathcal{O}\{l\} = q^l \, q\dim \mathcal{O}; \qquad \cdot[s]: \quad \text{height shift by } s;$$



$$m: \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases} \qquad \Delta: \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

That's a Frobenius Algebra! And a (1+1)-dimensional TQFT!

Example: $\quad q^{-2} + 1 + q^2 - q^6 \quad \xrightarrow[(\text{with } (n_+, n_-) = (3,0))]{\cdot(-1)^{n_-} q^{n_+ - 2n_-}} \quad q + q^3 + q^5 - q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$ if $\xi_j = \star$) $\qquad = \quad [\![\bigcirc]\!] \quad \xrightarrow[(\text{with } (n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}} \quad \mathcal{C}(\bigcirc)$.

Theorem 1. The graded Euler characteristic of $\mathcal{C}(L)$ is $\hat{J}(L)$.

Theorem 2. The homology $\mathcal{H}(L)$ is a link invariant and thus so is $Kh_{\mathbb{F}}(L) := \sum_r t^r \, q\dim \mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$ over any field $\mathbb{F}$.

Theorem 3. $\mathcal{H}(\mathcal{C}(L))$ is strictly stronger than $\hat{J}(L)$: $\mathcal{H}(\mathcal{C}(\bar{5}_1)) \neq \mathcal{H}(\mathcal{C}(10_{132}))$ whereas $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$.

Conjecture 1. $Kh_{\mathbb{Q}}(L) = q^{s-1}\left(1 + q^2 + (1+tq^4)Kh'\right)$ and $Kh_{\mathbb{F}_2}(L) = q^{s-1}(1+q^2)\left(1 + (1+tq^2)Kh'\right)$ for even $s = s(L)$ and non-negative-coefficients laurent polynomial $Kh' = Kh'(L)$.

Conjecture 2. For alternating knots s is the signature and Kh' depends only on tq^2.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and DBN's
http://www.ma.huji.ac.il/~drorbn/papers/Categorification/.