FLAG VARIETIES
ASSIGNMENT 1
DUE FRIDAY OCTOBER 14

(1) Let $\underline{k} = (k_1, \dots, k_m)$ be natural numbers with $k_1 + \dots + k_m = n$. For any vector space V of dimension n, let $Fl(\underline{k}, V)$ denote the partial flag variety

$$\{0 \subseteq V_1 \subseteq \dots \subseteq V_m = V : \dim V_i = \dim V_{i-1} + k_i\}$$

  (a) How many B-orbits are there on $Fl(\underline{k}, \mathbb{C}^n)$? (As usual B denotes the subgroup of upper triangular matrices.) As usual, these B-orbits are called Schubert cells.
  (b) Find the dimension of the Schubert cells.
  (c) Describe the Schubert cells in terms of intersections with the standard flag.
  (d) Recall that if $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, then H_λ denotes the set of Hermitian matrices with eigenvalues $\lambda_1, \dots, \lambda_n$. Recall that if λ_i are distinct, then we constructed an isomorphism between H_λ and the full flag variety. For each $\underline{k}$, find some λ, such that there is a natural bijection $H_\lambda \cong Fl(\underline{k}, \mathbb{C}^n)$.
  (e) A special case of a partial flag variety is a Grassmannian. This corresponds to the choice $\underline{k} = (k, n-k)$. In this case, show that there are $\binom{n}{k}$ Schubert cells.

(2) Let $\lambda \in \mathbb{Z}^n_+$ be a dominant weight and let $V(\lambda) = \Gamma(Fl_n, \mathcal{O}(\lambda))^*$. Recall that in class we showed that $V(\lambda)$ is an irreducible representation of GL_n.
  (a) In class, we constructed a non-zero N-invariant section $s \in \Gamma(Fl_n, \mathcal{O}(\lambda))$. Show that this section has weight $(-\lambda_n, \dots, -\lambda_1)$.
  (b) If V is any representation of GL_n and μ is any weight, use the embedding $S_n \subset GL_n$ to prove that $V_\mu \cong V_{w\mu}$ for any $w \in S_n$.
  (c) Use the first two parts to deduce that $V(\lambda)_{w\lambda}$ is 1-dimensional for each $w \in S_n$.

Recall that we write

$$\lambda \geq \mu, \quad \text{if } \lambda - \mu \in Q_+ = \{\sum_{i=1}^{n-1} p_i \alpha_i : p_i \in \mathbb{N}\}$$

(where $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$). The following Lemma from representation theory is helpful.

Lemma 1. *If $\lambda, \mu \in \mathbb{Z}^n_+$ and $\mu \leq \lambda$, then $V(\lambda)_\mu \neq 0$.*

  (d) Use the lemma to find all $\lambda \in \mathbb{Z}^n_+$ such that the restriction map $\Gamma(Fl_n, \mathcal{O}(\lambda)) \to \Gamma(Fl_n^T, \mathcal{O}(\lambda))$ is an isomorphism.

(3) In this question, we consider $Fl_n \to \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathrm{Sym}^k \mathbb{C}^n)$, where the first map takes $V_\bullet$ to V_1 and the second map is the Veronese.
  (a) Show that the pullback of $\mathcal{O}(1)$ to Fl_n is the line bundle $\mathcal{O}(k\omega_1)$.

(b) Show that $\mathrm{Sym}^k \mathbb{C}^n$ is an irreducible representation of Fl_n and that $\mathrm{Sym}^k \mathbb{C}^n \cong V(k\omega_1)$. (You can use the Lemma mentioned above.)

(c) Use our results from class regarding line bundles on flag varieties to show that the homogeneous coordinate ring of $\mathbb{P}(\mathbb{C}^n)$ inside $\mathbb{P}(\mathrm{Sym}^k \mathbb{C}^n)$ is

$$\bigoplus_{r=0}^{\infty} (\mathrm{Sym}^{rk} \mathbb{C}^n)^*$$

(d) Deduce that the ideal of $\mathbb{P}(\mathbb{C}^n)$ in $\mathbb{P}(\mathrm{Sym}^k \mathbb{C}^n)$ is generated by

$$\frac{\binom{n+k-1}{k}\left(\binom{n+k-1}{k}+1\right)}{2} - \binom{n+2k-1}{2k}$$

quadratic equations.

(e) In the case when $n = 2$, show that this simplifies to $k(k-1)/2$. Find these equations explicitly in coordinates.

(f) Can you find the equations in coordinates for general n?

(4) In this question, we consider the flag variety $Fl(V)$ where V is an n-dimensional vector space over a finite field k with q elements. This flag variety is a finite set.

(a) Using the decomposition of $Fl(V)$ into Schubert cells, find $|Fl(V)|$. (Your answer should be a sum over the symmetric group.)

(b) For any m-dimensional vector space W over k, show that

$$|\mathbb{P}(W)| = \frac{q^m - 1}{q - 1}$$

We define $[m]_q := q^m - 1/q - 1$

(c) Use the previous question to show that

$$|Fl(V)| = [n]_q[n-1]_q \cdots [1]_q$$

This number is often called the q-analog of $n!$.