FLAG VARIETIES
ASSIGNMENT 2
DUE FRIDAY NOVEMBER 11

(1) Let $\underline{k} = (k_1, \ldots, k_m)$ be natural numbers with $k_1 + \cdots + k_m = n$. Recall the partial flag variety $Fl(\underline{k}, \mathbb{C}^n)$.
  (a) Show that
  $$T^*Fl(\underline{k}, \mathbb{C}^n) = \{(X, V_\bullet) : V_\bullet \in Fl(\underline{k}, V), X \in \mathcal{N}, XV_i \subseteq V_{i-1}\}$$
  (b) Show that $T^*Fl(\underline{k}, \mathbb{C}^n)$ is the resolution of a nilpotent orbit closure. Which one?
  (c) For any $X \in \mathcal{N}$, let $Fl(\underline{k})^X$ denote the fibre of $T^*Fl(\underline{k}, \mathbb{C}^n)$ over the point X. Give a bijection between the irreducible components of $Fl(\underline{k})^X$ and the set of semistandard Young tableaux of shape λ and content $\underline{k}$. (Having content $\underline{k}$ means that the tableau contains k_1 1s, k_2 2s, …, k_m ms.)

(2) Consider the algebra $R = \mathbb{C}[x_1, \ldots, x_n]/\langle \mathbb{C}[x_1, \ldots, x_n]_+^{S_n} \rangle$, the quotient of the polynomial ring by the positive degree invariant symmetric polynomials. As discussed in class $R \cong H^*(Fl_n)$. Without using this geometric fact, prove that R is isomorphic to the regular representation $\mathbb{C}[S_n]$ of S_n. (Hint: one way to do this is to compute the character of both representations.)

(3) Let λ be a partition of n and let X be the nilpotent Jordan form matrix given by λ. In class, for any row-strict tableau U of shape λ, we define a point $E_\bullet^U \in Fl_n^X$. Also, for any standard Young tableau U, we defined a subset $Y_U \subset Fl_n^X$.
  (a) For any U, show that there exists a Schubert cell in Fl_n, such that Y_U is the intersection of Fl_n^X with this Schubert cell.
  (b) Find all pairs U, V (where U is row-strict and V is standard) such that $E_\bullet^U \in \overline{Y_V}$.
  (c) For each U standard, find an action of $\mathbb{C}^\times$ on Fl_n^X such that Y_U is the attracting set $E_\bullet^U$ for this action.

(4) For each n, let C_n denote the set of crossingless matchings of n points on a line. There is a diagrammatic algebra, called the Temperley-Lieb algebra TL_n which has basis C_n^2 (see for example, the wikipedia page). The algebra TL_n usually depends on a parameter denoted δ. We will be concerned with the case $\delta = 2$. (Sometimes it is defined with a parameter q; $\delta = 2$ corresponds to $q = 1$.)
  (a) Show that there is an algebra map $\mathbb{C}[S_n] \to TL_n$ given by $s_i \mapsto U_i - 1$.
  (b) Let V_n be a vector space with basis C_n. Show that there is a natural action of TL_n on V_n. By (a), this gives us an action of S_n on V_n. Show that V_n is the irreducible representation of S_n corresponding to the partition (n, n).
  (c) Let $F_n = Fl_n^X$ denote the Springer fibre to the (n, n) nilpotent matrix X. Recall that for each $U \in C_n$, we have an irreducible component

$\overline{Y_U} \subset F_n$, described by

$$\overline{Y_U} = \{V_\bullet \in Fl_n^X : X^{-k}V_i = V_j \text{ whenever region } i \text{ and region } j$$
$$\text{are connected in } U \text{ and } j = i + 2k \}$$

Thus we get a vector space isomorphism $H_{top}(F_n) \cong V_n$ taking $[\overline{Y_U}]$ to U.

Recall the isomorphism $H(Z) = \mathbb{C}[S_n]$ and the action of $H(Z)$ on $H_{top}(F_n)$. Prove by a direct computation that the isomorphism $H_{top}(F_n) \cong V_n$ is S_n-equivariant.