MAT 247
ASSIGNMENT 9
DUE TUESDAY APRIL 5

(1) Let $V = \mathbb{Q}$ (i.e. the 1-dimensional vector space over the field of rational numbers). Let H be a non-zero bilinear form on V.
   (a) Show that we can find a basis v for V such that $[H]_v = \begin{bmatrix} n \end{bmatrix}$, where n is a square-free integer (a non-zero integer which is not divisible by the square of an integer bigger than 1).
   (b) Show that such an n is unique (in the same sense that the signature of symmetric bilinear form on a real vector space is unique).

(2) Let V be a vector space and let Ω be a symplectic form on V. Let L, M be two Lagrangian subspaces of V such that
$$L \cap M = 0.$$
   (a) Show that $V = L \oplus M$.
   (b) Define a linear map $T : M \to L^*$ by setting $T(v)$ to be the linear functional defined by $(T(v))(w) = \Omega(v, w)$. Prove that T is an isomorphism.
   (c) Use the above results to find a symplectic basis
$$q_1, \ldots, q_n, p_1, \ldots, p_n$$
   for V such that $q_1, \ldots, q_n \in L$ and $p_1, \ldots, p_n \in M$.

(3) Let $V, \langle , \rangle$ be a complex inner product space. We can regard V as a real vector space by just considering scalar multiplication by elements of $\mathbb{R}$. Define a real bilinear form Ω on V by
$$\Omega(v, w) = Re(\langle v, iw \rangle),$$
where Re denotes the real part of a complex number.
   (a) Show that Ω is a real symplectic form on V.
   (b) Let $v_1, \ldots, v_n$ be a orthonormal basis for V (regarded as a complex inner product space). Show that
$$v_1, \ldots, v_n, iv_1, \ldots, iv_n$$
   is a symplectic basis for V (regarded as a real vector space with symplectic form Ω).

(4) Recall that if $T : V \to W$ is a linear map, then there is a linear map $T^* : W^* \to V^*$ which is defined by

$$(T^*(\beta))(v) = \beta(Tv),$$

for $\beta \in W^*$ and $v \in V$. Applying this reasoning twice, we see that there is a linear map $(T^*)^* : (V^*)^* \to (W^*)^*$.

In class, for each vector space V, we defined the isomorphism $\psi_V : V \to (V^*)^*$ by setting $\psi_V(v)$ to be the linear functional on V^* defined by

$$(\psi_V(v))(\alpha) = \alpha(v).$$

for each $\alpha \in V^*$.

Prove that for any two vector spaces V, W and any linear map $T : V \to W$, we have $(T^*)^*\psi_V = \psi_W T$.

(This is the sense in which ψ is “natural”.)